Economies with Asymmetrically Informed Agents: the Concept of Limit Information *

MARAKULIN V. M.†
Institute of Mathematics, Russian Academy of Sciences, Novosibirsk

Abstract

In this paper, a new concept of "limit information" is introduced and studied for Arrow–Debreu type economies with asymmetrically informed agents. The concept is based on the so-called contractual approach that presumes that agents meet and form coalitions, where the concurrent exchange of commodities and information is realized. So, in the course of a natural exchange process, the agents' information is repeatedly transformed and accumulated, and agents learn and achieve limit information. It is proved that, for a monotone information sharing rule, limit information is unique, i.e., it does not depend of the chain of coalitions implemented in the process of interaction between economic agents.

Key words: exchange economy, contract, asymmetric information, core.
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Introduction

Information plays an important role in economic analysis. We face it not only in the case of individual decision making, but also in the case of the workings of markets and of the whole economy. In particular, information undoubtedly plays an important role in the allocation of resources sold through the system of markets. However, in a classic version of economic modelling, the issue of information has not been considered. For example, in the context of the Arrow — Debreu — McKenzie model, it is implicitly assumed that the whole economic activity takes place as if in a separate time period, and agents possess sufficient information on the values of economic variables to make their rational decisions, and transactions are carried out for infinitesimal time, etc. (e.g., see Aliprantis, et al. (1989)). In reality, however, players in the economy have to make their decisions, given a shortage of information, and, at the same time, they are asymmetrically informed on states of the world.[1] Thus, information asymmetry is a feature of the real economy disregarded by classic theory.

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†Valeriy M. MARAKULIN, Institute of Mathematics, Russian Academy of Sciences, 4 Acad. Koptyug avenue, Novosibirsk, Russia, 630090, e-mail: marakul@math.nsc.ru

[1]States of the world or states of nature are any factors affecting economic variables, primarily wealth. States of nature can include the state of health of an individual, acts of God (natural disasters), exchange rates, etc.

The development of models and concepts of solution (equilibrium, core etc.) that properly take into account the incompleteness and asymmetry of the information that individuals possess mainly started in the second half of 1970s. Here, first of all, it is worth mentioning the work of Radner (Radner, 1968), where a model economy with differential information arises, an appropriate generalization of Walrasian expectations equilibrium (WEE) is formulated, and the existence theorem is proved. Further, we note Wilson's seminal paper (Wilson, 1978), where the concepts of coarse core and fine core are introduced, several interesting specific examples are studied, and original concepts of equilibrium are suggested, which have no specific names in the current literature, see Glycopantis, Yannelis (2004)). This work has drawn attention of economists, and it has had a profound impact on the development of economic theory in the context of models with differential information. One should mention another work of Radner (1978), where it is assumed that individuals can extract information from the distribution of prices, and the important concept of rational expectations equilibrium (REE) is introduced. A specific feature of this concept is that the consumer is assumed to maximize the *conditional* expectation of utility,[2] given initial information and additional information. In the subsequent research, many authors (see Differential information economies (2004), Economic Theory 18 (2001)) have developed the theory of differential information economy along many directions. Among them, different concepts of core have been developed and refined,[3] the existence and relationships with well-known concepts of equilibrium have been studied, important issues of incentive compatibility[4] and of the implementation of a core allocation as an equilibrium in some strategic game, which is constructed from the initial model economy, have been examined, and a series of other directions have been considered; the most complete surveys of the literature are presented in Differential information economies (2004), Schwalbe (1999).

In the mentioned Wilson's paper (Wilson, 1978), the (non-formalized) concept of communication system appears, as a medium of information transmission from some agents to others. In the subsequent papers of Allen (Allen, 1991, 1994), the concept is generalized and formalized as an information rule, a map that transforms the information available to members of a coalitions into a form that can be used by them to dominate the current allocation. Thus, Allen formalizes the most general way to define the core in the economy with information asymmetry. Applying different rules in the context of the same model (hence, different measurability conditions on a dominating allocation), one can obtain any of well-known concepts of core (coarse core, fine core, private core etc.). Schwalbe (1999) develops the approach of Allen and introduces the concept of *maximal information*, which is the most complete information that an agent can obtain being a

[2]It is this feature that constitutes a distinction of kind between this concept and that of Walrasian expectation equilibrium (WEE), also known as Radner equilibrium, which should not be confused with REE, see Radner (1968).

[3]In models with asymmetrically informed agents, the measurability of the final information distribution and distributions that are permitted to dominate this distribution by coalitions plays an important role. It is important that the measurability may *differ*, but it is always given by the initial information distribution, which is given by partitions of possible future states of the world. E.g., in the case of the final distribution, it may be joint information, i.e., the supremum of agents' private information; in the case of domination it may be common information, i.e., the infimum of the private information of members of a coalition. In this way, the concept of fine core is obtained. There are many ways to arrange choices, which implies multiple concepts of core, some of them may lead to the empty set.

[4]This means the absence of incentives to misinform other agents about the state of the world. Potentially, this may be the case, since one agent may exactly know the state of the world (after its realization) while others cannot distinguish it from some other states.

member of all possible coalitions that he/she joints with his/her initial information. It is the measurability with respect to maximal information that is assumed by Schwalbe as a basis of conditions imposed on the final resource allocation. At the same time, coalitions can dominate only through intra-coalition allocations measurable with respect to the information derived by the rule from the initial information, as Allen assumes. However, both authors do not clarify what happens to information. It would seem that, joining different coalitions, agents extract fresh information. In doing so, they form the final allocation. Then agents forget everything and try to dominate this allocation... There is some flaw in this view of (intra-model) economic activity. In our view, this approach is not quite satisfactory. However, can one suggest any constructive approach instead? Economic theory focuses on the analysis of the final resource allocation with equilibrium properties. But in reality, and from the perspective of the *contractual approach*, this allocation is an outcome of multiple transactions of exchange among economic agents. But, at the same time, the exchange of information is also going on; and not each transactions of exchange is feasible, one of the reasons being information shortcoming.

The idea of the contractual approach is by no means new in economic theory. Seemingly, it goes back to the classic results of Edgeworth (1881). The notion of contract as a barter trade in commodities then appears in the work of other authors (even though it has not been elaborated), including Russian ones: Polterovich (1970), and Makarov (1982). Kozyrev (1981, 1982) suggests the important concept of partially broken contracts, which provides an alternative description of Walrasian equilibrium. Finally, the author of this paper (Marakulin, 2003, 2006) has developed the base of the theory of barter contracts,[5] which can be viewed as an augmentation of classic ideas on the working of the market economy.

In our view, the contractual approach is a rather convenient tool for the modelling of environments with asymmetrically informed agents. Indeed, in the contractual approach framework, the concept of admissibility of a contract plays an important role. Clearly, a reasonable description of real transactions requires at least clarifying what kind of contracts can be concluded. The set of concluded contracts and the initial resource allocation yield a resource allocation, which may have or may not have certain properties of stability. In particular, the theory of barter contracts studies contractual allocations, properly contractual allocations, perfectly contractual allocations etc. (Marakulin, 2003), and contractual processes resulting in such allocations (Marakulin, 2006). In this paper, we are interested in information characteristics of contractual processes and theoretical conclusions resulting from them. When extending the contractual approach to models with asymmetric information, a key requirement is the measurability of a contract, as a part of the conditions of its feasibility. This implies that the contract is measurable with respect to the algebra of events that is given by the distribution of information among economic agents. However, the information distribution in the economic environment is not static. It is being changed over time in the course of economic interaction among agents. Note that information can be transformed in the same way as Allen assumes, by an exogenous rule. But, unlike Allen and Schwalbe, we assume that, when a transaction is closed, information does not disappear, but, in contrast, it is accumulated.

The conceptual part of this paper describes the concept of *limit information*. An essential distinction of limit information from maximal information is that the concept

[5]In the first paper, the contractual approach is used to introduce a core into models of incomplete markets in a well-defined way. The second paper tries to justify the working of the economy in disequilibrium and the subsequent transition to an equilibrium.

of limit information takes account of agents' ability *to learn* in the course of exchange of commodities and information. Moreover, in reality, an agent does not usually have access to maximal information and has to use less complete limit information, that is, the information that he/she manages to find out. Formally, limit information is an outcome of information exchange implemented through a chain of economic interactions among individuals in the context of intra-coalition exchange of commodities and concurrent exchange of information. We show by examples that, generally, different "chains" may lead to different distributions of information, which cannot be improved by subsequent transactions of exchange. Therefore, one of the primary issues that require clarification is the identification of conditions on information exchange under which limit information is uniquely defined. It is proved that if a rule of information sharing is monotone, then limit information is unique and, hence, the concept is well-defined in this case.

The paper is organized as follows. The first section is devoted to main concepts, notation, and the description of an exchange economy with asymmetric information. The second section considers issues of extending the contractual approach to model economies with asymmetric information. Here the concept of limit information is introduced and discussed, and the uniqueness theorem is proved.

1 A model economy with asymmetric information

Economies with asymmetric information that have been by now described in the literature can be roughly divided into two classes, the guideline for this division being the definition of information. In economic theory, information has been considered from two points of view. On the one hand, it has been observed that information has certain characteristics of a commodity (in usual, physical, sense) and thus is similar to other commodities. That is, it can be exchanged, bought or sold in corresponding information markets. On the other hand, the information available to an agent can be seen as a characteristic of this agent, which is similar to his initial endowment of goods and his preferences. Below a model is described that takes into account both aspect of information. The description is conducted in accordance with the terminology employed in Schwalbe (1999), where a current view on asymmetric awareness of agents is provided.

1.1 Agents and their information

Consider an exchange economy with a finite set of agents $\mathcal{I} = \{1, 2, \dots, n\}$, where $i \in \mathcal{I}$ denote an agent. A specific feature of the model is explicitly introduced information on states of the world (nature) that is available to agents, where information generally differs across agents and can be changed in the course of their economic activity.

Information is modelled as follows. Consider a measurable space $(\Omega, P(\Omega))$ of events of nature. Within this section and the next section, the set Ω is assumed to be *finite*. Elements $\omega \in \Omega$ are called states of nature or states of the world, as well as elementary events. Here $P(\Omega)$ is a partition of the set Ω. Recall that a *partition* of an (arbitrary) set Ω is any collection of its pairwise disjoint subsets whose union is Ω. Denote by P^* the universe of all partitions of the set Ω, and let Ω^* be a partition that consists of one-element subsets of Ω. The information available to agent $i \in \mathcal{I}$ is characterized by a partition $P_i \in P^*$. Informally, if an element of a partition consists of several states of nature, this means that the agent cannot distinguish among these states. In this way, the agent's ability to distinguish events is described.

Let $P_i(\omega)$ be the element of a partition P_i that contains a state $\omega \in \Omega$. A version of information $P \in P^*$ is said to be *better (finer)* than a version of information $P' \in P^*$ if every element of P is a subset of some element of P', i.e. $P(\omega) \subseteq P'(\omega), \forall\, \omega \in \Omega$. Informally, one version of information is finer than another one if it can better distinguish elementary states of nature. The relation "*is finer than*" is a partial order relation on the set P^* of all informational partitions of Ω, and further it will be denoted by

$$P \succeq P' \iff P \;\; \textbf{\textit{is finer than}} \;\; P'.$$

It is easily seen that $\succeq$ induces the lattice on the set P^* of all partitions, i.e., for any finite set of partitions, there exist a supremum and an infimum.

An ordered collection (tuple) of individualized partitions $\mathbb{P} = (P_i)_{i\in\mathcal{I}}$ is called an ***informational structure of the economy***. ¡

The relation $\succeq$, which is defined on informational partitions, induces a partial order relation on the set of all informational structures defined as follows:

$$\mathbb{P} \succeq \mathbb{P}' \iff P_i \succeq P_i', \quad \forall\, i \in \mathcal{I}.$$

The relation $\succeq$ will be also applied to coalition informational structures in order to compare different informational provisions of a certain coalition.

Below well-known definitions are formulated (e.g., see Schwalbe (1999)):

Definition 1.1 *If $P_i = \Omega^*$, , then an informational partition P_i of an agent i is called* ***perfect*** *or* **complete**. *An informational structure of an economy is called* ***perfect*** *if each agent's version of information is perfect.*

Definition 1.2 *An informational structure $(P_i)_{i\in\mathcal{I}}$ is called* ***symmetric*** *if $P_i = P_j$, $\forall\, i, j \in \mathcal{I}$.*

Definition 1.3 *An informational structure $(P_i)_{i\in\mathcal{I}}$ is called* ***asymmetric*** *if $P_i \neq P_j$ for some $i, j \in \mathcal{I}$, $i \neq j$.*

Clearly, the definition of perfect information is applicable to a single agent, and the concepts of symmetric and asymmetric information are defined for, at least, two agents. Note also that if information is perfect, then it is symmetric; but if an information structure is asymmetric, then it cannot be perfect. Generally, the reverse is not true.

1.2 Commodities, agents, and consumption plans

Denote by $L = \{1, 2, \ldots, l\}$ the list of physical commodities (goods) in the economy. Thus, the space of physical commodities is $E = \mathbb{R}^L$, i.e. an l-dimensional Euclidean space. Contingent on events, individuals can consume different bundles of physical (contingent) commodities. The space of contingent commodities is the set $\mathbf{Map}(\Omega, \mathbb{R}^L)$ of all maps from the space of elementary events Ω into $\mathbb{R}^L$.

Let $P \in P^*$ be some partition of Ω. A function f whose domain is Ω is called *P-measurable* if it constant[6] on elements of the partition P. For $P \in P^*$, introduce a set

$$\mathbf{Map}_P(\Omega, \mathbb{R}^L) := \{f : \Omega \to \mathbb{R}^L \mid f|_{P(\omega)} = const\},$$

[6] For a finite algebra of events, this definition is equivalent to the standard one.

which is the subspace of all P-measurable maps of $\mathbf{Map}(\Omega, \mathbb{R}^L)$.

A distinctive feature of Schwalbe's approach (Schwalbe, 1999) is that information is considered as a component of the consumption bundle; uncertainty being modelled as follows: a good, in addition to a place of consumption and a time of consumption, is specified by a state of the world (nature).

Since physical commodities are considered being related to states of the world, it is natural to postulate that the space of admissible consumption bundles depends on information. Accordingly, *the generalized space of the goods* $\mathbf{Map}(\Omega, \mathbb{R}^L) \times P^*$ arises. In this space, goods are presented by ordered pairs whose components are a vector of physical contingent commodities and information. Thus, information is included into the definition of the good.

In order to illustrate this idea, consider the following example. Let there be two different states of nature ω_1, ω_2:

ω_1 — agent i is infected,

ω_2 — agent i is not infected.

Consider two versions of information that can be available to agent i:

$P_i = \{\{\omega_1, \}, \{\omega_2\}\}$ — the agent distinguishes these states.

$P_i' = \{\{\omega_1, \omega_2\}\}$ — the agent does not distinguish these states.

Suppose that there is a medicine that can cure the agent's disease. It is obvious that, given P_i, the drug is not the same good as it is, given P_i'.

Thus, each player $i \in \mathcal{I}$ is characterized with his/her initial endowment $(e_i, {P_i}^0) \in \mathbf{Map}(\Omega, \mathbb{R}^L) \times P^*$, where $e_i \in \mathbf{Map}(\Omega, \mathbb{R}^L)$ denotes agent i's initial endowment of commodities, and ${P_i}^0$ denotes his/her initial stock of information (a partition of Ω). For the sake of simplicity, we will assume that individuals are able to consume only non-negative quantities of physical commodities, and, in addition, $e_i \geq 0$, $\forall\, i \in \mathcal{I}$.

The consumption set of agent $i \in \mathcal{I}$ is defined as follows:

$$X_i := \{(x, P) \in \mathbf{Map}(\Omega, \mathbb{R}^L_+) \times P^* \mid x - e_i \in \mathbf{Map}_P(\Omega, \mathbb{R}^L)\}.$$

It is obvious that $(e_i, {P_i}^0) \in X_i$ (since $e_i - e_i = 0$ is compatible with any information, $X_i \neq \emptyset$).

The consumption plan of agent i is an ordered pair (x_i, P_i), where the first component $x_i \in \mathbf{Map}(\Omega, \mathbb{R}^L_+)$ is his/her consumption plan, and the second component denotes the information available to the agent. The consumption set of the agent consists of all consumption plans that are realised by a web of contracts compatible with information P_i. Note that all this is true for each P_i. That is, a plan (x, P) for an arbitrary P belongs to the consumption set if $(x - e_i)$ is a constant map on elements of the set P. The agent cannot consume goods independently of his/her information. That is, if the agent does not distinguish two states of the world, then a consumption plan realised by some web of contract and differing in these states of nature does not belong to the consumption set.

A collection of the agents' consumption plans $((x_i, P_i))_{i \in \mathcal{I}}$, where $(x_i, P_i) \in X_i$, is called ***a state of the economy, or an allocation***.

It is important to note that, in all early models, information is regarded just as a restriction on the consumption set, whereas in this model information is a component of the consumption set.

1.3 Coalitions and rules of information sharing

Nonempty subsets of $\mathcal{I}$ are called *coalitions*. Let $C \subseteq 2^{\mathcal{I}} \setminus \{\emptyset\}$ be the set of all admissible coalitions. Only coalitions from C can be formed in order to implement any plans (with respect to an exchange of commodities, information etc.). Further, without loss of generality, we will consider that $C := 2^{\mathcal{I}} \setminus \{\emptyset\}$, i.e., that any coalition is allowed.

It is assumed that members of a coalition can exchange their experience, i.e., they can modify their private information. Initial endowments of agents are reallocated on basis of this modified information. Consider the process of information exchange among agents who joins a coalition.

Information exchange is described by an information rule that is a given model parameter. Each member of a coalition possesses some information, therefore, saying on the information available to the members of the coalition, we can imagine it as a collection of information partitions. The information rule associates each coalition and each collection of (private) information of this coalition with the new collection of information partitions. Note that component i of the new collection denotes the information that member i of the coalition can use within the framework of intra-coalition activity.

Definition 1.4 ***An information rule of a coalition*** *$S \subseteq \mathcal{I}$ is a map $k_S : (P^*)^S \to (P^*)^S$ satisfying $k_{\{i\}}(P) = P$, $\forall\, i \in \mathcal{I}$, $P \in P^*$.*

An information ***rule of an economy*** *is $(2^{|\mathcal{I}|} - 1)$-tuple of maps $k = (k_S)_{S \in C}$, where each coalition S is put into correspondence with its information rule k_S.*

Informally, this can be expressed as follows. After joining a coalition S with information P_i, member i of the coalition S ($i \in S$) has access to information $P_i' = k_S^i((P_j)_{j\in S})$, which is component i, a projection, of the value of the map k_S at the "point" $P_S = (P_i)_{i\in S}$. Note that information is not modified in one-element coalition, since there is nobody to exchange with.

In further analysis, it will be sometimes convenient to consider the map k_S, $S \in C$ to be defined on the set of all informational structures. In that case, without further reminding, we will always assume that the information of agents outside of this coalition stays unchanged.

Thus, the result of applying an information rule is new information, which can be used by agents to reallocate initial endowments within the coalition.

Definition 1.5 *An information rule $k = (k_S)_{S\in C}$ is called* ***a rule of information sharing*** *if $k_S((P_i)_{i\in S}) \succeq (P_i)_{i\in S}$ for each $(P_i)_{i\in\mathcal{I}}$, $\forall\, S \in C$.*

Let $\Re$ be the set of all rules of information sharing. The partial order relation $\succeq$ ("is better than"), being defined on the set of all partitions of Ω, induces the following binary relation on $\Re$:

Let k and κ be two arbitrary rules of information sharing. We say that k *is better than* κ if $k_S^i((P_j)_{j\in S})$ is better than $\kappa_S^i((P_j)_{j\in S})$ for any coalition $S \subseteq \mathcal{I}$, any agent $i \in S$, and any collection $P_S = (P_i)_{i\in S}$. For short, suppose that:

$$k \ \textbf{\textit{is better than}}\ \kappa \iff k \succeq \kappa.$$

The relation $\succeq$ is reflexive, transitive, but not complete. Thus, the set $\Re$ is a partially ordered set with respect to $\succeq$.[7]

[7] It is clear that $\succeq$ can be applied not only to rules of information sharing, but also to the broader class of all information rules.

Further we give a series of well-known definitions (see Schwalbe (1999)).

An information rule is called ***symmetric*** if, for any coalition $S \subseteq \mathcal{I}$ and any two agents $i, j \in S$, $k_S^i((P_j)_{j\in S}) = k_S^j((P_i)_{i\in S})$ holds. That is, as a result of applying this information rule, each members of a coalition possesses the same information.

A rule of information sharing is called ***dense*** if, for any $i \in \mathcal{I}$ and any S, T such that $i \in S \subset T$, $k_T^i((P_j)_{j\in T}) \succeq k_S^i((P_j)_{j\in S})$ holds. In other words, this means that if a coalition expands, then the information of any member of the coalition cannot worsen.

A rule of information sharing is called ***bounded*** if, for any $i \in \mathcal{I}$ and any S, it is true that $k_{\mathcal{I}}^i((P_j)_{j\in\mathcal{I}}) \succeq k_S^i((P_j)_{j\in S})$.

The definition of the bounded information rule can be obtained from the definition of dense information rule when the latter is applied to the coalition of all agents. Obviously, if an information rule is dense, then it is bounded, but the converse is not true.

Further we mention several examples presented in the literature, which illustrates the concept of the information rule:

Example 1.1 The coarsest information rule $k^c := (k_S^c)_{S\in C}$ is defined as follows: $k_S^c((P_i)_{i\in S}) = (P_S^c)_{i\in S}$, where $P_S^c := \underset{S}{\wedge} P_i$ for any coalition S.

Here $\underset{S}{\wedge} P_i$ denotes the finest partition derived from $(P_i)_{i\in S}$, given that, for any information P_i, each element of P_i is a subset of some element of the partition $\underset{S}{\wedge} P_i$. Thus, $\underset{S}{\wedge} P_i$ is the infimum of P_i, $i \in S$. □

The coarsest information rule describes a setting, where members of a coalition cannot communicate, and this rule is not a rule of information sharing. Here a transaction of exchange can only result in allocations that are measurable with respect to the coarsest information. In other words, when a transaction among members of a coalition occurs, all agents as a whole must understand the problem at hand, that is, they must distinguish events relevant to the transaction. Note that the foregoing is not a requirement for agents to forget their initial information when joining the coalition. However, it may be the case that, concluding the contract, agents fail to use their initial information in full. The coarsest information rule is symmetric, but it is nor dense, neither bounded.

Example 1.2 The finest information rule $k^f := (k_S^f)_{S\in C}$ is defined as follows: $k_S^f((P_i)_{i\in S}) = (P_S^f)_{i\in S}$, where $P_S^f := \underset{i\in S}{\vee} P_i$, for any coalition S, where $\underset{S}{\vee} P_i$ is the supremum of information partitions: $(\underset{S}{\vee} P_i)(\omega) := \underset{i\in S}{\cap} P_i(\omega)\ \forall\, \omega \in \Omega$. □

The finest information rule describes a setting, where members of a coalition can exchange their information in full. This information is generated as follows. Basing on the information of each member of the coalition, we construct the partition that is the coarsest one of partitions that are finer than any individual partition of a member of the coalition. The result will be the best information. In this case, the information, which each member of the coalition uses to reallocate resources, will be not worse than his/her individual information. Note that the finest rule of sharing information is symmetric, dense, and bounded.

Example 1.3 A private information rule $k^p := (k_S^p)_{S\in C}$ is defined as follows: $k_S^p((P_i)_{i\in S}) = (P_i)_{i\in S}$ for each coalition S, i.e., $k_S^p = Id$ for each coalition S. □

Thus, this rule does not modify the information available to members of a coalition: for each agent, his/her information remains unchanged.

Example 1.4 A special case of the information rule is the zero information rule. Irrespective to the initial information of agents, this rule assigns information $P_i = \{\Omega\}$ to each member of a coalition. □

1.4 Feasible allocations

The definition of allocation is of importance when we consider the core of an economy. Certainly, feasible states of the economy, which we will call allocations, must be, at least, physically feasible. However, if we consider an economy with asymmetric information, then the definition of feasibility must also take into account the initial information available to agents and the rule of information sharing applied in this model.

In the literature, there are two most important definitions of (feasible) allocation. The first one has been suggested by Yannelis, who says that an allocation is feasible if it is compatible with the initial information of agents. Thus, according to Yannelis, the possibility of information exchange among agents is not taken into consideration. The second definition originates in the work of Allen (Allen, 1991). Defining feasible allocations, she takes into account not only the information that is initially available to agents but also information exchange. Allen has considered allocations to be feasible if they are measurable with respect to the information available to each agent in the economy, that is, with respect to the coarsest information. Both of these definitions have clear shortcomings. Obviously, the definition of Yannelis implies that the set of feasible allocations does not depend on the rule of information sharing in any way, though it is intuitively clear that there must be some relation. The definition of Allen is excessively rigid because it may be that agents cannot use any information. In this case, it may be that the only feasible allocation is given by the initial endowments of agents.

Being motivated by these reasons, Schwalbe (Schwalbe, 1999) introduces the concept of *maximal information*, which he uses later on to define the concept of feasible allocation.

Definition 1.6 ***The maximal information*** *of agent* $i \in \mathcal{I}$ *with respect to a rule* $k = (k_S)_{S \in C}$ *is* $P_i^{max} := \bigvee_{S \in C} k_S^i((P_j{}^0)_{j \in S})$.

Thus, the maximal information is the information that the agent would obtain if he/she were able to join all coalitions simultaneously.

Employing the concept of maximal information, one can define feasible allocations.

Definition 1.7 *An allocation* $((x_i, P_i))_{i \in \mathcal{I}} \in \mathbf{Map}(\Omega, \mathbb{R}_+^L) \times P^*$ *is called* ***feasible*** *if*

(i) $\sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i$,

(ii) $(x_i - e_i) \in \mathbf{Map}_{P_i^{max}}(\Omega, \mathbb{R}^L)\ \forall\, i \in \mathcal{I}$.

Here the first condition reflects the fact of physical feasibility of the allocation; whereas the second one reflects its informational feasibility, since it requires the measurability of the gross contract (net trade) implementing the allocation.

In the next section, the new concept of limit information is introduced, which makes it possible to consider a new concept of feasible allocation. The distinction consists of other requirements to measurability from **(ii)**, now we require measurability with respect to a limit information partition.

2 Limit information

2.1 The concept of limit information

In the previous section, we describe the exchange economy model, where agents are informed in different ways. The model assumes the exchange of both physical commodities and information. In subsection 1.4, the issue of feasible allocations of the form $((x_i, P_i))_{i\in\mathcal{I}}$ has been discussed. Here the concept of feasible allocation, along with the usual balancing condition on consumption bundles, $\sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i$, incorporates the condition of measurability of maps $(x_i - e_i)$ with respect to maximum information, that is, $(x_i - e_i) \in \mathbf{Map}_{P_i^{max}}(\Omega, \mathbb{R}^L)\ \forall\, i \in \mathcal{I}$. In other words, the only things that matter in the model are the initial allocation and the final one. The chain of exchanges of commodities and information that makes it possible the transition from the initial allocation to the final one remains "off-camera", and all attention is exclusively focused on the final allocation of commodities. However, the contractual approach implies an explicit sequence of transactions of exchange. Therefore, further we will consider precisely transactions of concurrent exchange of commodities and information, i.e., contracts. In analyzing contracts, we focus primarily on the exchange of information. But in doing so, we will not forget that any transaction also involves the exchange of physical commodities, since it is the allocation of resources that is the main target of the further analysis.

It is appropriate to explain briefly what leads agents to disclose their information. To do so, turn to the concept of measurability of the contract. In essence, the contract is a collection $v = (v_i)_{i\in\mathcal{I}}$ of maps $v_i : \Omega \to \mathbb{R}^L$ such that $\sum_{i\in\mathcal{I}} v_i = 0$. Besides, commodity flows v_i must satisfy the condition of measurability with respect to the algebra of events corresponding to the awareness of individuals. Formally, we can consider the measurability of the contract with respect to any information. However, looking informally, only two cases of measurability are of the most importance: the measurability of the contract with respect to individual information of agents and its measurability with respect to the coarsest information. The first case implies that, when concluding a contract, each agent should know what he/she receives; the second case implies that every agents in the economy should know what other agents receive. It is clear that, in both cases, the finer the information is, the weaker the condition of measurability is, and, hence, the broader the set of feasible allocations is. Therefore, in order to extend the space of feasible allocation, agents agree about the exchange of information.

One of our purposes is to understand what kind of agents' informational provision can, in theory, result from transactions of exchange. Here there may be different cases and earlier we have presented some examples of information rules and the information resulting from the application of these rules (the finest information, the coarsest information, private information etc.). In particular, we have introduced the concept of maximal information P_i^{max}, $i \in \mathcal{I}$, which can be used for any information rule. In our view, a drawback of this concept is that it is far from being realistic, since it is unclear why the agent, which repeatedly joins new coalitions, can use only his/her initial information over and over again. Actually, the agent probably possesses more complete information, which he/she has obtained from the experience of closed transactions. A concept that takes into account the agents' capabilities of keeping in mind and of learning is called *limit information*. The precise definition is given below.

Definition 2.1 *The information (of an agent) is called **limit** if there exists a sequence of coalitions such that this information is attained at the last stage of exchange among agents* *and remains unchanged (i.e., it cannot be improved) in the course of any subsequent intra-coalition information exchange.*

Note that the concept of limit information implies that, in each act of sharing information, agents can use the information that they have received earlier from preceding exchanges. Thus, information can be accumulated, and agents can learn.[8]

Formally, information $\mathbb{P}^{lim} = (P_i^{lim})_{i\in\mathcal{I}}$ is limit if there exists a (finite) chain of coalition $S_1, S_2, \ldots, S_m \subseteq \mathcal{I}$, $m \in \mathbb{N}$ such that

(i) $P_i^{\xi} = k^i_{S_\xi}((P_j^{\xi-1})_{j\in S_\xi})$, $i \in S_\xi$, $P_i^{\xi} = P_i^{\xi-1}$, $i \in \mathcal{I} \setminus S_\xi$, $\xi = 1, 2, \ldots, m$;

(ii) $\forall\, S \subseteq \mathcal{I}$, $\forall\, i \in S$, $k^i_S((P_j^m)_{j\in S}) = P_i^m = P_i^{lim}$.

Thus, it is assumed that individuals participate in the repetitive process of information exchange that is given by an information rule and a certain sequence of coalition. Here information is initially exchanged among members of the coalition S_1 in accordance with the information rule k_{S_1}, whereas the information available to agents outside of this coalition is not changed at this stage. At the second stage, information is exchanged only among members of the coalition S_2, and this exchange may involve agents that modified their information at the first stage. But in this case they participate in the exchange of information, being equipped with fresh information. And so on, until information exchange is ended.

The concept of limit information gives rise to a number of questions with respect to its properties. They include the following:

a) Is limit information unique? If it is not, then under which conditions is the rule of information sharing (or just the information rule) associated with unique limit information?

b) Is any actual difference between the two formally different concepts, those of limit information and of maximal information? It might be the case that limit information and maximal information, being defined in different ways, coincide.

Further we will consider examples that illustrate that limit information indeed differs from maximal information. Then we will clarify the issue of uniqueness.

2.2 Feasible allocations and uniqueness

Recall that, at each stage, both information and commodities are exchanged, resulting eventually in a final allocation. Therefore, it would be appropriate to define feasible allocations, using the concept of limit information.

Denote by P_i^{lim} the limit information of agent i. If n is a number of agents in the economy, then the n-tuple $\mathbb{P}^{lim} = (P_1^{lim}, P_2^{lim}, \ldots, P_n^{lim})$ is called *limit informational structure of the economy.*

Definition 2.2 *An allocation $((x_i, P_i))_{i\in\mathcal{I}}$ is called limiting if:*

(i) $\sum\limits_{i\in\mathcal{I}} x_i = \sum\limits_{i\in\mathcal{I}} e_i$,

(ii) $(x_i - e_i) \in \mathbf{Map}_{P_i^{lim}}(\Omega, \mathbb{R}^L)\ \forall\, i \in \mathcal{I}$.

[8]This is rather “eductive learning” and “rational learning” than “adaptive learning”.

Note that, seemingly, the concept of limiting allocation can be reasonably used only in the case of uniqueness of limit information. However, the example described below 2.1 illustrates that uniqueness does not always take place. In other words, limit information may depend on a chain of coalitions through which this information is formed. Besides, Example 2.1, alongside with Example 2.2, shows that limit information may be finer and may be coarser than maximal information.[9]

Example 2.1 Consider 4 states of nature $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and 3 agents with their initial information:

$$P_1^0 = \{\omega_1, \omega_2, \omega_3, \omega_4\}, \quad P_2^0 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \quad P_3^0 = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}.$$

Let $S_1 = \{1, 2\}, S_2 = \{1, 3\}, S_3 = \{2, 3\}, S_4 = \{1, 2, 3\}$ denote the coalitions.

The rule of sharing information for a coalition $S \subseteq \mathcal{I}$***:*** *In the coalition S the act of sharing information occurs if S contains an agent and there is an element $\widetilde{\omega} \subset \Omega$ of his/her informational partition such that* ***any other*** *member of the coalition has an element of his/her partition that includes the* ***whole*** *event $\widetilde{\omega}$. Then all agents of the coalition S become able to distinguish the event $\widetilde{\omega}$ and, if possible, the next act of information sharing occurs (within the iteration) etc. If these conditions are not satisfied, then information is not exchanged.*

Note that if an individual is able to distinguish events $\widetilde{\omega} \subset \Omega$, $\widetilde{\widetilde{\omega}} \subset \Omega$, and, in addition, $\widetilde{\omega} \subset \widetilde{\widetilde{\omega}}$, then this individual is also able to distinguish the event $\widetilde{\widetilde{\omega}} \setminus \widetilde{\omega}$.

Informally, the described rule implies the following. Suppose that there exists a member of a coalition S that realizes that he/she is able to distinguish an event $\widetilde{\omega}$ while the rest of members S are able to distinguish broader events than $\widetilde{\omega}$. Then this agent initiates information sharing. The agent transmits the ability to distinguish this events to the rest of agents. Transmitting his/her knowledge to the others, this agent expects to receive additional information from them, immediately or, possibly, in the subsequent iterations of information sharing, being a member of other coalitions.

Suppose that at first agent 1 joined the coalition $S_1 = \{1, 2\}$. Then his/her information P_1^1 (the superscript is the iteration's number) equals $\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$. Obviously, this information cannot be improved in the course of any subsequent intra-coalition exchange and, hence, is limit.

Consider another possible sequence of coalitions. Suppose that at first agent 1 joined the coalition $S_2 = \{1, 3\}$. Then his/her information P_1^1 is assigned the value $\{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$. Again, this information is limit.

Finally, note that $P_1^{max} = \{\{\omega_1\}, \{\omega_3\}, \{\omega_2\}, \{\omega_4\}\}$, i.e., in this case the maximal information of agent 1 is finer than any version of its limit information. □

The following example shows that limit information may be finer than maximal information.

Example 2.2 Consider 8 states of nature $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}$ and 3 agents with their initial information:

$P_1^0 = \{\{\omega_2, \omega_3\}, \{\omega_1, \omega_4, \omega_6\}, \{\omega_5, \omega_7, \omega_8\}\}$,

[9] The rule of information sharing and examples 2.1, 2.2 have been constructed in Kadyrova (2003).

$P_2^0 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5, \omega_6\}, \{\omega_7, \omega_8\}\}$,
$P_3^0 = \{\{\omega_1, \omega_2, \omega_3, \omega_8\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_7\}\}$.

Suppose that the rule of sharing information is the same as in the previous example and coalitions are denoted as earlier.

Then consider maximal information $\mathbb{P}^{max}$. By definition, for agent 1 we have:

$$P_1^{max} = \bigvee_{S \ni 1} k_S^1(P^0) = k_{S_1}^1(P_1^0, P_2^0) \vee k_{S_2}^1(P_1^0, P_3^0) \vee k_{S_4}^1(P_1^0, P_2^0, P_3^0),$$

where
$k_{S_1}^1(P_1^0, P_2^0) = \{\{\omega_2, \omega_3\}, \{\omega_1, \omega_4, \omega_6\}, \{\omega_5\}, \{\omega_7, \omega_8\}\}$,
$k_{S_2}^1(P_1^0, P_3^0) = \{\{\omega_2, \omega_3\}, \{\omega_1, \omega_4, \omega_6\}, \{\omega_5, \omega_7, \omega_8\}\}$,
$k_{S_4}^1(P_1^0, P_2^0, P_3^0) = \{\{\omega_2, \omega_3\}, \{\omega_1, \omega_4, \omega_6\}, \{\omega_5, \omega_7, \omega_8\}\}$.
As a result, we have:

$$P_1^{max} = \{\{\omega_2, \omega_3\}, \{\omega_1, \omega_4, \omega_6\}, \{\omega_5\}, \{\omega_7, \omega_8\}\}.$$

Next we find out a possible version of limit information. Suppose that at first the coalition $S_1 = \{1, 2\}$ is "active". We obtain:
$P_1^1 = k_{S_1}^1(P_1^0, P_2^0) = \{\{\omega_2, \omega_3\}, \{\omega_1, \omega_4, \omega_6\}, \{\omega_5\}, \{\omega_7, \omega_8\}\}$,
$P_2^1 = k_{S_1}^2(P_1^0, P_2^0) = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_6\}, \{\omega_5\}, \{\omega_7, \omega_8\}\}$,
$P_3^1 = P_3^0 = \{\{\omega_1, \omega_2, \omega_3, \omega_8\}, \{\omega_4, \omega_5\}, \{\omega_6, \omega_7\}\}$.
Suppose that next the coalition $S_2 = \{1, 3\}$ is "active". We have:
$P_1^2 = k_{S_2}^1(P_1^1, P_3^1) = \{\{\omega_1, \omega_6\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_7, \omega_8\}\}$,
$P_2^2 = P_2^1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_6\}, \{\omega_5\}, \{\omega_7, \omega_8\}\}$,
$P_3^2 = k_{S_2}^3(P_1^1, P_3^1) = \{\{\omega_1, \omega_8\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6, \omega_7\}\}$.
Suppose that the coalition is $S_1 = \{1, 2\}$ "active" again. We have:
$P_1^3 = k_{S_1}^1(P_1^2, P_2^2) = \{\{\omega_1, \omega_6\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_7, \omega_8\}\}$,
$P_2^3 = k_{S_1}^2(P_1^2, P_2^2) = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_6\}, \{\omega_4\}, \{\omega_5\}, \{\omega_7, \omega_8\}\}$,
$P_3^3 = P_3^2 = \{\{\omega_1, \omega_8\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6, \omega_7\}\}$.
It is easily seen that any further process of intra-coalition information sharing cannot continue. As a result, for agent 1 we have:

$$P_1^{lim} = \{\{\omega_2, \omega_3\}, \{\omega_1, \omega_6\}, \{\omega_4\}, \{\omega_5\}, \{\omega_7, \omega_8\}\}.$$

Thus, the limit information P_1^{lim} of agent 1 is finer than his/her maximal information P_1^{max}. Therefore, in this example limit information is coarser than the finest information (i.e., complete information) and is finer than maximal information, i.e., lies in between them. □

Certainly, there are some ways of applying limit information even in the case of its non-uniqueness. E.g., we can consider the following way of defining the core. A coalition dominates an allocation if, after the breaking of all contracts, there is a (new) chain of coalitions whose members conclude mutually beneficial contracts and exchange information in such a way that the final allocation is "better" for member of this coalition. However, in this case it is not clear what has happened to agents' information. Do agents forget everything? But how can they compare the new allocation with the old one then?.. In addition, we can consider maximal limit information that is determined as the supremum of all versions of limit information... However, economic interpretation remains unclear again.

2.3 A criterium of uniqueness

We have noted in the previous section that, in order to apply the concept of limit information in a well-defined way, we need to reveal uniqueness conditions, i.e., the conditions under which it does not depend on how coalitions in the sequence of transactions of exchange are ordered. Below we formulate a sufficient condition for uniqueness of limit information. It turns out that, for each *monotone* rule of information sharing, limit information is unique.

Definition 2.3 *An information rule $k = (k_S)_{S \in C}$ is* ***monotone*** *if, for each coalition $S \in C$ and any informational structures, from $\mathbb{P} \succeq \mathbb{P}'$ it follows that $k_S(\mathbb{P}) \succeq k_S(\mathbb{P}')$.*

In other words, an information rule is monotone if, for each coalition, it preserves the natural partial order relation on the set of all informational structures. Here it is appropriate to note an important point. A nonmonotone rule need not have a property that, for some $\mathbb{P} \succ \mathbb{P}'$ and some coalition S, it is true that $k_S(\mathbb{P}) \prec k_S(\mathbb{P}')$. The relation $\succeq$ is only *partial* ordering on the set of informational structures, i.e., there exist noncomparable structures. If a rule is nonmonotone, then, for some coalition S and a pair of informational structures, it is true that $\mathbb{P} \succ \mathbb{P}'$ & $k_S(\mathbb{P}) \not\succeq k_S(\mathbb{P}')$. This is not equivalent to the previous statement. This situation is illustrated by the rule from Example 2.1, where $\mathbb{P}' = k_{S_1}(\mathbb{P}^0) \succ \mathbb{P}^0$ but both $k_{S_2}(\mathbb{P}') \not\succeq k_{S_2}(\mathbb{P}^0)$ and $k_{S_2}(\mathbb{P}') \not\preceq k_{S_2}(\mathbb{P}^0)$ hold, i.e., the structures that are obtained by the rule of information sharing are noncomparable.

Theorem 2.1 (of uniqueness) *If a rule of* ***information sharing*** *$k \in \Re$ is* ***monotone****, then the limit informational structure $\mathbb{P}^{lim}$ that is obtained by the rule k is unique.*

Proof. Let $\alpha = \{S_1, S_2, \ldots, S_r\}$ and $\beta = \{T_1, T_2, \ldots, T_q\}$ be two sequences of coalitions, which generate, in accordance with Definition 2.1, two versions of limit information, $\mathbb{P}^{lim}_\alpha$ and $\mathbb{P}^{lim}_\beta$. We will show that $\mathbb{P}^{lim}_\alpha = \mathbb{P}^{lim}_\beta$.

Indeed, by Definition 1.5, for any rule of information sharing, we have

$$\mathbb{P}^0 \preceq k_{S_1}(\mathbb{P}^0) = \mathbb{P}^1_\alpha \preceq k_{S_2}(\mathbb{P}^1_\alpha) = \mathbb{P}^2_\alpha \preceq \ldots \preceq k_{S_r}(\mathbb{P}^{r-1}_\alpha) = \mathbb{P}^r_\alpha = \mathbb{P}^{lim}_\alpha \quad \Rightarrow \quad \mathbb{P}^0 \preceq \mathbb{P}^{lim}_\alpha.$$

We sequentially apply the monotonicity of the rule and the definition of limit information. By doing so, we conclude that

$$\mathbb{P}^1_\beta = k_{T_1}(\mathbb{P}^0) \preceq k_{T_1}(\mathbb{P}^{lim}_\alpha) = \mathbb{P}^{lim}_\alpha \quad \Rightarrow \quad \mathbb{P}^2_\beta = k_{T_2}(\mathbb{P}^1_\beta) \preceq k_{T_2}(\mathbb{P}^{lim}_\alpha) = \mathbb{P}^{lim}_\alpha \ldots \Rightarrow$$

$$\mathbb{P}^{q-1}_\beta = k_{T_{q-1}}(\mathbb{P}^{q-2}_\beta) \preceq k_{T_{q-1}}(\mathbb{P}^{lim}_\alpha) = \mathbb{P}^{lim}_\alpha \quad \Rightarrow \quad \mathbb{P}^{lim}_\beta \preceq \mathbb{P}^{lim}_\alpha.$$

By using similar reasoning, first with respect to the sequence β and then with respect to α, we find out that $\mathbb{P}^{lim}_\beta \succeq \mathbb{P}^{lim}_\alpha$. Hence, $\mathbb{P}^{lim}_\beta = \mathbb{P}^{lim}_\alpha$. Theorem 2.1 is proved. □

Corollary 2.1 *If a rule of information sharing $k \in \Re$ is monotone, then* ***the limit*** *informational structure $\mathbb{P}^{lim}$* ***is finer than the maximal one****, i.e., $\mathbb{P}^{lim} \succeq \mathbb{P}^{max}$.*

Proof. It is sufficient to show that, for any coalition $S \subseteq \mathcal{I}$ and any $i \in \mathcal{I}$, $k^i_S((P_j{}^0)_{j \in S}) \preceq \mathbb{P}^{lim}_i$ holds. However, this is the case, since if the coalition S is added to the beginning of any sequence of coalitions implementing limit information, then the new chain also implements limit information. At the same time, as the rule of information sharing is repeatedly applied, information can become only finer at each stage and it automatically becomes limit after the last stage. Further, in order to complete this proof, we need to use Definition 1.6:

$$P_i^{max} := \bigvee_{S \in C} k_S^i((P_j{}^0)_{j \in S}) \preceq \mathbb{P}_i^{lim}.$$

□

It is obvious that limit information exists (since Ω is finite). But how can we find it in a certain model? The answer to this question is given by the following algorithm.[10]

Consider an arbitrary finite sequence of coalitions

$$\pi = \{S_1, S_2, \ldots, S_m\}$$

such that it contains *all* possible coalitions. Let k be an arbitrary rule of information sharing that is monotone.

An algorithm generating limit information: *We subsequently apply rules* k_{S_ξ}, $\xi = 1, 2, \ldots, m$, *beginning with* $\mathbb{P}^0$, *the order is given by* π. *Once the exchange of information has been initiated and then completed within some coalition, we will return to the beginning of the sequence* π. *After that we apply the same operation to the new initial information that is obtained as a result of information sharing at the previous stage. The operation of the algorithm is ended, once the sequence* π *is completely passed through with no exchange of information initiated within any coalition. Thus, using a finite number of iterations, we attain limit information* $\mathbb{P}^{lim}$.

Due to Theorem 2.1 it is obvious that this algorithm does generate limit information. Certainly, one can also suggest other algorithms similar to the just described one.

Which coalitions can guarantee the attainment of limit information? This question matters, since in reality not each coalition can be realized and, in addition, the intensity of information exchange may differ across coalitions. Certainly, the answer to this question depends on specific features of a rule of information sharing. E.g., in the case of a *monotone and dense* rule (see section 1.3) of information sharing, it is obvious that, doing on its own, the coalition of *all* agents can attain the limit informational structure. However, is such a coalition viable? And what information can agents receive by joining this coalition? It seems to us that, in small-size coalitions, information sharing should be more intense than it is in "large" ones, though a broader coalition potentially possesses more information.

Reasoning may be as follows. Suppose that there is a monotone rule of information sharing k. It is clear that limit information depends on the list C of admissible coalitions, since in theory different lists may implement different limit structures, and a broader list implements finer information. Therefore, we can formulate the following question: Which list of coalitions that is minimal by inclusion implements the same limit information, as does the complete list of all coalitions? Obviously, such lists exist. We can also ask a

[10] This algorithm and a proof of the uniqueness theorem, which is based on it and is different from the proof presented above, have been constructed by M. Predtechensky, who received his bachelor degree under my supervision in the Department of Mechanics and Mathematics, Novosibirsk State University, in 2005.

simpler question: When can pairwise coalitions implement (full-value) limit information? An incomplete answer to this question is given by the following consideration. Let $S \in C$ be a coalition such that for each $i \in S$ the following condition is satisfied:

$$k^i_S(({P_j}^0)_{j\in S}) \preceq \bigvee\{k^i_T(({P_j}^0)_{j\in T}) \mid T \neq S,\ T \in C\}, \tag{2.1}$$

Alternatively, a stronger condition may be satisfied.

$$k^i_S(({P_j}^0)_{j\in S}) \preceq \bigvee\{k^i_T(({P_j}^0)_{j\in T}) \mid T \neq S,\ T \subset S,\ T \in C\}. \tag{2.2}$$

Then it is easily seen that the coalition S can be excluded from the list of coalitions specifying the limit information. Indeed, in this case one can replace each occurrence of the coalition S in a chain of coalition that implements limit information by somehow ordered collection of all other coalitions. In this way one obtains a chain implementing the same limit informational structure. However, this chain does not include coalition S. Hopefully, by this way of consecutive exclusion of coalitions from the list of admissible ones, we will narrow this list substantially. It is easily seen that in the case of (2.2) one will reach the list consisting of pairwise coalitions. Further we can apply the condition (2.1) again, narrowing the list even more... By this way, in general, one needs not obtain a minimal list,[11] but we manage to shorten it essentially. At this point we finish our discussion of the concept of limit information.

3 An Infinite Set of Elementary Events

In this section we consider a generalization of the simplest model economy with asymmetrically informed agents to the case of an *infinite* set Ω.

If Ω is infinite, then modelling information of an agent via a partition of Ω is incorrect, since many interesting cases remain out of the analysis. Therefore, further the information of an individual is understood as the entire σ-algebra of events that the individual can distinguish. Thus, let $\mathcal{A}_i = \mathcal{A}_i(\Omega)$ be a σ-algebra of events distinguishable by agent i and $(\Omega, \mathcal{A}_i(\Omega))$ be his/her measurable space, i.e., the information of individual i. The informational structure of the economy is the tuple $\mathbb{A} = (\mathcal{A}_i)_{i\in\mathcal{I}}$.

All basic concepts considered earlier can be directly extended to the case of an infinite set of elementary events, without principal difficulties. Therefore, further we point out only some cases that requires special agreements.

The partial order relation $\succeq$ given on the set of all partitions of Ω is extended to the case of an infinite set Ω by the following rule:

$$\mathcal{A} \succeq \mathcal{A}' \iff \mathcal{A} \supseteq \mathcal{A}'.$$

It is easily seen that, in the case of a finite set of states of nature, this definition is equivalent to the relation "is finer than" on partitions. Similar to the finite case, the relation $\succeq$ is extended to informational structures:

$$\mathbb{A} \succeq \mathbb{A}' \iff \mathcal{A}_i \supseteq \mathcal{A}'_i \ \ \forall\, i \in \mathcal{I}.$$

Now we can consider such concepts as a rule of information sharing and monotone information rules. These concepts are absolutely similar to the case of a finite set of

[11] To achieve this on the right-hand side of (2.1) we should use not only coalitions, but also chains of coalitions that apply the rule of information sharing consecutively.

states of nature. Moreover, the concept of feasible allocation is also extended directly from the finite case to the infinite one: certainly, we need to apply the correct concept of measurable map. Further we consider the concept of limit information.

Informally, in the case of an infinite number of states of nature, the concept of limit information still makes sense. However, its formalization needs to be adjusted. Namely, in the finite case, limit information is constructed through a finite chain of coalitions. In the infinite case, we need to apply an infinite sequence such that, after *each occurrence* of any *coalition* in the sequence, *this coalition will appears further in the sequence.* In other words, the process of exchange of commodities and information continues infinitely, and if some exchange of information is possible in any coalition, then this coalition must be implemented.

Definition 3.1 *Let* $\pi = (S_1, S_2, \ldots, S_\xi, \ldots)_{\xi=1,2,\ldots}$ *be sequence of coalitions such that, for each* $\xi = 1, 2, \ldots$ *and each* $S \subseteq \mathcal{I},\ S \neq \emptyset$, *there exists* $\xi' \geq \xi$ *such that* $S_{\xi'} = S$. *Then limit information with respect to* π *is found from conditions:*

(i) $\mathcal{A}_i^\xi = k^i_{S_\xi}((\mathcal{A}_j^{\xi-1})_{j\in S_\xi}),\ i \in S_\xi, \quad \mathcal{A}_i^\xi = \mathcal{A}_i^{\xi-1},\ i \in \mathcal{I}\setminus S_\xi,\ \ \xi = 1, 2, \ldots,$

(ii) $\mathcal{A}_i^{lim} = \bigvee\limits_{\xi=1}^{\infty} \mathcal{A}_i^\xi,\ \forall\, i \in \mathcal{I}.$

Note that this definition implicitly assumes that the rule $k = (k_s)_{S\in C}$ is a rule of information sharing,[12], i.e., information can be only augmented in the course of intra-coalition interaction. It is clear that this definition of limit information generally depends on a chosen sequence of coalitions. However, the following theorem shows that, just as in the finite case, limit information is unique for a monotone rule of information sharing.

Theorem 3.1 (of uniqueness) *If a rule of* ***information sharing*** $k \in \Re$ *is* ***monotone***, *then the limit information structure* $\mathbb{A}^{lim} = (\mathcal{A}_i^{lim})_{i\in\mathcal{I}}$ *is unique, i.e., it does not depend on the chosen sequence that implements it.*

Proof. Consider two arbitrary sequence of coalitions that generate, by definition, 2.1, two versions of limit information, $\mathbb{P}_\alpha^{lim}$ and $\mathbb{P}_\beta^{lim}$, respectively:

$$\alpha = \{S_1, S_2, \ldots, S_\xi, \ldots\}, \qquad \beta = \{T_1, T_2, \ldots, T_\xi, \ldots\}.$$

Let us show that $\mathbb{P}_\alpha^{lim} = \mathbb{P}_\beta^{lim}$.

Indeed, by Definition 3.1, the sequences α and β are such that there exist monotone maps $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ of the positive integers $\mathbb{N}$ into itself such that

$$T_\xi = S_{f(\xi)} \quad \& \quad S_\xi = T_{g(\xi)} \qquad \forall\, \xi = 1, 2, \ldots.$$

Besides, from the monotonicity of the maps, we have $f(\xi+1) > f(\xi)$ and $g(\xi+1) > g(\xi)$, $\forall\, \xi = 1, 2, \ldots$ In other words, each of the sequences can be embedded into the other one, the initial ordering of coalitions being preserved.

Further, by applying repeatedly the definition of the rule of information sharing, we conclude that $\mathbb{A}^0 \preceq \mathbb{A}_\alpha^{f(1)-1}$ whence, due to the monotonicity of the rule, we have

$$\mathbb{A}_\beta^1 = k_{T_1}(\mathbb{A}^0) \preceq k_{T_1}(\mathbb{A}_\alpha^{f(1)-1}) = k_{S_{f(1)}}(\mathbb{A}_\alpha^{f(1)-1}) = \mathbb{A}_\alpha^{f(1)}.$$

[12]The concept of limit information can be also introduced for a general information rule, by $\mathcal{A}_i^{lim} = \bigwedge\limits_{m=1}^{\infty} \bigvee\limits_{\xi=m}^{\infty} \mathcal{A}_i^\xi$, $\forall\, i \in \mathcal{I}$ in **(ii)**, but, in the framework of this paper, there is no need to do so...

Similarly, $\mathbb{A}_\alpha^{f(1)} \preceq \mathbb{A}_\alpha^{f(2)-1} \Rightarrow \mathbb{A}_\beta^1 \preceq \mathbb{A}_\alpha^{f(2)-1}$ whence

$$\mathbb{A}_\beta^2 = k_{T_2}(\mathbb{A}_\beta^1) \preceq k_{T_2}(\mathbb{A}_\alpha^{f(2)-1}) = k_{S_{f(2)}}(\mathbb{A}_\alpha^{f(2)-1}) = \mathbb{A}_\alpha^{f(2)}$$

etc. As a result, we conclude that

$$\mathbb{A}_\beta^\xi \preceq \mathbb{A}_\alpha^{f(\xi)}, \quad \forall\, \xi = 1, 2, \ldots$$

Consequently, due to (**ii**) from the Definition 3.1, it must be the case that

$$\mathbb{A}_\beta^{lim} = \bigvee_{\xi=1}^{\infty} \mathbb{A}_\beta^\xi \preceq \bigvee_{\eta=1}^{\infty} \mathbb{A}_\alpha^\eta = \mathbb{A}_\alpha^{lim}.$$

Finally, replacing α with β in this reasoning and applying the embedding g, we conclude similarly that $\mathbb{A}_\beta^{lim} \succeq \mathbb{A}_\alpha^{lim}$ and, hence, $\mathbb{A}_\beta^{lim} = \mathbb{A}_\alpha^{lim}$. Theorem 3.1 is proved. □

Conclusions

The paper has suggested the new concept of limit information, which can be considered as an alternative to the concept of maximal information. We have proved that any monotone rule of information sharing implies the uniqueness of limit information. Possible non-uniqueness of limit information and its distinction from maximal information is illustrated by the supportive examples. It remains unclear whether the monotonicity of the rule of information sharing is a necessary condition for the uniqueness of limit information. It is not clear under which conditions maximal information coincides with limit information.

We are convinced that the suggested concept of limit information will be useful in constructing new economic models that take account of informational aspects and in studying well-known ones. In particular, this is applicable to the model that is described in Schwalbe (1999).

References

Allen, B. (1991). Market games with asymmetric information and nontransferable utility: representation results and the core// *University of Pensilvanya, CARESS Working Paper*, #91-09.

Allen, B. (1991). Transferable utility market games with asymmetric information: representation results and the core// *University of Pensilvanya, CARESS Working Paper*, #91-16.

Allen, B. (1994). Market games with asymmetric information: the core with finitely many states of the world// *in*: Models and Experiments in Risk and Rationality, eds. Munier, B. and M. J. Machina. Kluwer Academic Press.

Aliprantis, C. D., Brown, D. J., Burkinshaw, O. (1989). Existence and optimality of competitive equilibria, Berlin/New-York: Springer-Verlag.

Aliprantis, C. ed. (2001). Economic Theory, volume 18. Berlin/Heidelberg: Springer–Verlag.

Glycopantis, D., Yannelis, N. C. (2004). Differential information economies (collected papers). Berlin/Heidelberg: Springer–Verlag.

Glycopantis, D., Yannelis, N. C. (2004). Information, efficiency, and the core of an economy: comments on Wilson's paper// *in*: Differential Information Economies, 65–71, *Springer–Verlag*.

Kadyrova A. (2003). Contractual approach in models with asymmetrically informed agents// bachelor thesis, *Novosibirsk State University*.

Kozyrev, A. N. (1982). The contractual and fully contractual allocations in an abstract economy//*Institute of mathematics SB RAS USSR*, Novosibirsk. Preprint No. 7, 44 p. (in Russian)

Kozyrev, A. N. (1981). The stable systems of contracts in an exchange economy//*Optimization*, 29(44), 66–78 (the issue of Institute of mathematics SB AS USSR, in Russian).

Makarov, V. L. (1982). Economic equilibrium: the existence and extremal properties//*Sovremennye problemy matematiki. Itogi nauki i tehniki*, vol. 19, 23–58 (in Russian).

Marakulin, V. M. (2003). Contracts and domination in incomplete markets// *Economic Education and Research Consortium*. Working Paper Series, No 02/04, 104 p.

Marakulin, V. M. (2006). On convergence of contractual trajectories in pure exchange economies// *Economic Education and Research Consortium*. Working Paper Series, No 06/07, 78 p.

Polterovich, V. M. (1970). Mathematical models of resourse allocations// *Central Economic-mathematical Institute*, Moscow, 107 p. (in Russian).

Radner, R. (1968). Competitive equilibrium under uncertainty// *Econometrica*, 36, 31–58.

Radner, R. (1978). Rational expectation equilibrium: generic existence and information revealed by prices// *Econometrica*, 47, 655–678.

Schwalbe, U. (1999). The core of economies with asymmetric information. Berlin/Heidelberg/New York: Springer–Verlag.

Wilson, R. (1978). Information, efficiency, and the core of an economy// *Econometrica*, 46, 807–816.