# Evaluating the Degree of Manipulability of Certain Aggregation Procedures under Multiple Choices*

Fuad T. Aleskerov,[1] Daniel S. Karabekyan,[2]
Remzi M. Sanver,[3] and Vyacheslav I. Yakuba[4]

The problem of manipulation in voting is studied in the case of multi-valued choice. Well-known and new schemes of preferences extension enabling one to compare all possible social choices under an arbitrary number of alternatives are presented. The known indices of degree and efficiency of manipulation are described, and new indices are introduced. These preference extension schemes and indices are used for computer-aided evaluation of the degree and efficiency of manipulation of the known voting procedures allowing multi-valued choices. The results obtained are presented for five voting rules.

Key words: manipulation, multiple choice, voting, social choice
JEL classification: D7

## 1. Introduction

The problem of voting manipulation lies in the fact that the voter can achieve for herself a more advantageous social decision by intentional deviation from her true preference. The theoretical studies of the problem of manipulation were initiated in (Gibbard, 1973; Satterthwaite, 1975) where it was proved that any reasonable – namely, non-dictatorial – procedure is not protected against manipulation by the participants.

The question of the degree of manipulability of the voting schemes was for the first time posed in (Aleskerov, Kurbanov, 1999; Kelly, 1993). However, evaluation of the degree of manipulability in practice is a computationally complex problem, and some assumptions are made to simplify it. Consideration of the manipulation problem under single-valued choice by introducing a condition for elimination of incomparability is the most important one. This condition lies in that a single alternative is selected by a predefined rule from the set of winning alternatives. For example, alphabetic elimination of incomparability is discussed in (Aleskerov, Kurbanov, 1999). This method is most popular, but gives rise to many problems such as disparity of the participants (the voters preferring the first alphabetic alternatives

---

* The authors would like to thank A.A. Zaslavskii and the anonymous referee for useful remarks.
The work of F. Aleskerov and D. Karabekyan was supported in part by Scientific Foundation of SU HSE, project no. 08-04-0008 and the Russian Foundation for Basic Research, project no. 08-01-00039a. The work of V. Yakuba was supported in part by the Russian Foundation for Basic Research, project no. 08-01-00039a.
[1] State University, Higher School of Economics, Moscow, Russia; Trapeznikov Institute of Control Sciences, RAS, Moscow, Russia
[2] State University, Higher School of Economics, Moscow, Russia, danyakar@gmail.com
[3] Bilgi University, Istanbul, Turkey
[4] Trapeznikov Institute of Control Sciences, RAS, Moscow, Russia

are in more advantageous position), which can substantially distort evaluation of the level of manipulability. The method of (Pritchard, Wilson, 2005) which may be regarded as a softer condition for elimination of incomparability implies that in final multiple choices all alternatives are equiprobable and the winning alternative is selected randomly.

The purpose of the present paper lies in studying manipulability of the aggregation procedures under multiple choices, analyzing the existing schemes of preference extension, and developing new schemes which enable one to compare every possible social choice for any number of alternatives, adapting the existing and creating new indices for estimation of manipulability. This study allows one to carry out computer-aided evaluation of the degree of manipulability and manipulation efficiency of the aggregation procedures in the case of multiple choices.

Section 2 introduces the main notations and defines manipulation for the case of multiple choices. Methods of preference extension are described in Section 3, and Section 4 describes the manipulability indices. Section 5 defines the rules of voting. The scheme of calculations is presented in Section 6, and Section 7 discusses the results obtained.

## 2. Main notions and definitions

The problem of manipulation is considered using the notions and formulation given in (Aleskerov, Kurbanov, 1999). There is a set $A$ of $m$ alternatives $(m > 2)$. The set of all nonempty subsets of $A$ is denoted by $\mathbf{A} = 2^A \setminus \{\varnothing\}$. Each participant $i$ from the finite set $N = \{1,...,n\}, \quad (n > 1)$, has the preference $P_i$ on the set of alternatives from $A$ and an extended preference $EP_i$ on the set $\mathbf{A}$.

The preference $P_i$ is a linear order, if it satisfies the following conditions:

- irreflexivity ($\forall x \in A,\ x\overline{P}x$);
- transitivity ($\forall x, y, z \in A \ \ xPy$ and $yPz \Rightarrow xPz$); and
- connectedness ($\forall x, y \in A \ \ x \neq y$ either $xPy$ or $yPx$).

The vector $\vec{P}$ consisting of the preferences of $n$ participants is called the profile. The collective choice is generated from the profile $\vec{P}$ and is an element of the set $\mathbf{A}$. We denote by $\mathbf{L}$ the set of all linear orders on $A$. Therefore, the collective choice function is the map $F \ :\ \mathbf{L}^n \to \mathbf{A}$.

Now we define manipulation for the case of multiple choices. Let

$$\vec{P} = \{P_1, \ldots, P_i, \ldots, P_n\}$$

be the profile of the true preferences of the participants, whereas

$$\overrightarrow{P_{-i}} = \{P_1, \ldots, P_{i-1}, P_i', P_{i+1}, \ldots, P_n\}$$

be the profile where all participants but the $i$-th one declare their true preferences. For these profiles $\vec{P}$ and $\overrightarrow{P_{-i}}$, the generated choices are denoted, respectively, by $C(\vec{P})$ and $C(\overrightarrow{P_{-i}})$. Manipulation will be said to take place if there exists a participant $i$ and her preference $P_i'$ such that $C(\overrightarrow{P_{-i}})EP_iC(\vec{P})$ holds, where $EP_i$ is the extended preference of the $i$-th participant defined completely by $P_i$. Stated differently, we assume that from the point of view of the $i$-th participant the collective choice under the distorted preferences is more preferable than the one obtained from the true preferences.

In what follows, we use by way of example the “standard preferences” by which we mean the preferences where the alternatives are ordered alphabetically or according to their numeration, that is, the preferences $P_i$ like $x_1P_ix_2P_ix_3P_i \ldots P_ix_n$ or $aP_ibP_icP_i \ldots P_iz$.

## 3. Methods of preference extension

Some properties describing preferences on the set of alternatives can be found in (Barbera, 1977; Gärdenfors, 1976; Kannai, Peleg, 1984; Kelly, 1977). Yet they allow one to compare only certain types of sets. We describe below several existing and new methods of preference extension which can be classified with the groups of lexicographic, probabilistic, and rank-averaging methods. The latter method *per se* does not allow one to compare every possible set of alternatives and, therefore, needs additional constraints. Eight types of such constraints are proposed which provide various extended preferences in the form of linear order.

### *Lexicographic Methods*

#### Leximin

This method of extension was proposed in (Pattanaik, Peleg, 1984) but we use it in the form given in (Sanver, Ozyurt, forthcoming). The method relies on comparing the alternatives in two collective choices that are worst in terms of the original preference. If the worst alternatives coincide, it is required to compare the next to the worst alternatives, and so on. If further comparison is impossible, that is, one of the collective choices is a subset of the other, then the more wide set is preferred to the smaller one.

In formal terms, this method is as follows. The leximin principle $EP_i$ is used to construct extended preferences for all $i \in N$ and preferences $P_i \in \mathbf{L}$ on the basis of the following algorithm.

Two collective choices $X, Y \in \mathbf{A}$ are compared.

1. Let $|X| = |Y| = k$, where $k \in \{1, \ldots, m-1\}$. We order the elements of collective choice in the descending order of preference $P_i$, that is, $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$, where $x_j P_i x_{j+1}$ и $y_j P_i y_{j+1}$ $\forall j \in \{1, \ldots, k-1\}$. The relation $X\ EP_i\ Y$ takes place iff $x_h P_i y_h$ for the greatest $h \in \{1, \ldots, k\}$, such that $x_h \neq y_h$.
2. Let $|X| \neq |Y|$. We order the elements of the collective choice in the ascending order of the preference $P_i$, that is, $X = \{x_1, \ldots, x_{|X|}\}$ и $Y = \{y_1, \ldots, y_{|Y|}\}$, where $x_{j+1} P_i x_j$ $\forall j \in \{1, \ldots, |X|-1\}$ and $y_{j+1} P_i y_j$ $\forall j \in \{1, \ldots, |Y|-1\}$. Now, two cases are possible:
   a. $x_h = y_h$ $\forall h \in \{1, \ldots, \min\{|X|, |Y|\}\}$, that is, one set is a subset of the other. Then, as was already noticed, the more wide set is more preferable. Stated differently, $X\ EP_i\ Y$ iff $|X| > |Y|$
   b. $\exists h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$. In this case, $X\ EP_i\ Y$ iff $x_h P_i y_h$ for the smallest $h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$.

For example, for the case of three alternatives and standard preferences, the extended preferences constructed using the leximin principle look as:

$$\{a\} EP_i \{a,b\} EP_i \{b\} EP_i \{a,c\} EP_i \{a,b,c\} EP_i \{b,c\} EP_i \{c\}.$$

**Leximax**

This method is similar to leximin, but the best elements are compared of two collective choices. If they are equal, then the next to the best elements are compared, and so on. If further comparison is impossible, then one of the sets is a subset of the other set, and the narrower set is preferred to the larger one.

We also propose a formal description of this method according to (Sanver, Ozyurt, forthcoming). For all $i \in N$ and preferences $P_i \in \mathbf{L}$, the extended preferences are constructed according to the principle of leximax $EP_i$ as follows.

Two collective choices $X, Y \in \mathbf{A}$ are compared.

1. Let $|X| = |Y| = k$, where $k \in \{1, \ldots, m-1\}$. We order the elements of the collective choice in the descending order of the preference $P_i$, that is, $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$, where $\forall j \in \{1, \ldots, k-1\}$ $x_j P_i x_{j+1}$ and $y_j P_i y_{j+1}$. The relation $X\ EP_i\ Y$ takes place iff $x_h P_i y_h$ for the least $h \in \{1, \ldots, k\}$ such that $x_h \neq y_h$.
2. Let $|X| \neq |Y|$. Similarly, we order the elements of the collective choice in the descending order of the preference $P_i$, that is, $X = \{x_1, \ldots, x_{|X|}\}$ and $Y = \{y_1, \ldots, y_{|Y|}\}$, where $\forall j \in \{1, \ldots, |X|-1\}$ $x_j P_i x_{j+1}$ and $\forall j \in \{1, \ldots, |Y|-1\}$ $y_j P_i y_{j+1}$. Now, two cases are possible:
   a. $x_h = y_h \quad \forall h \in \{1, \ldots, \min\{|X|, |Y|\}\}$, that is, one set is a subset of the other. Stated differently, $X\ EP_i\ Y$ iff $|X| < |Y|$;
   b. $\exists h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$. In this case, $X\ EP_i\ Y$ iff $x_h P_i y_h$ for the smallest $h \in \{1, \ldots, \min\{|X|, |Y|\}\}$ for which $x_h \neq y_h$.

For example, for the case of three alternatives and standard preferences, the leximax-based extended preferences are as follows:

$$\{a\} EP_i \{a,b\} EP_i \{a,b,c\} EP_i \{a,c\} EP_i \{b\} EP_i \{b,c\} EP_i \{c\}.$$

Both methods of extension enable us to compare every possible set.

## *Probabilistic Methods*

These methods of alternative ordering differ from the leximin and leximax methods in that the voter is oriented not to the presence of an alternative in the collective choice, but rather to the probability that finally this alternative wins, that is, is selected as the result of using some method of eliminating multiplicity of choice, e.g. like tossing. Two methods are used: ordering by descending probability of the best alternative and ascending probability of the worst alternative.

### Ordering by Descending Probability of the Best Alternative

The method is based on element-wise comparison of two multiple choices. If different alternatives occupy the first places in the ordered sets corresponding to these choices, then the set with the best alternative is preferred. If identical alternatives occupy the first places, then that set is preferred where the probability of the final choice of the given alternative is higher. In the case of equal alternatives, relations of other elements are considered.

We discuss this method by an example. In the set $\{a,b,c\}$ the probability that the final choice includes the alternative $a$ is $\frac{1}{3}$ (the probabilities of victory of each alternative from one multiple choice are equal), and in the set $\{a,c\}$ the probability is $\frac{1}{2}$. Therefore, these two sets are related by $\{a,c\}EP_i\{a,b,c\}$ if the extended preferences are constructed using the principle of descending probability of the best alternative. In the general form, for the case of three alternatives and standard preferences, the extended preferences are as follows:

$$\{a\}EP_i\{a,b\}EP_i\{a,c\}EP_i\{a,b,c\}EP_i\{b\}EP_i\{b,c\}EP_i\{c\}$$

We present a formal description of this method. For all $i \in N$ and preferences $P_i \in \mathbf{L}$, the extended preferences $EP_i$ are constructed as follows.

We order the elements of the collective choice in the descending order of preferences $P_i$, that is, $X = \{x_1, \ldots, x_{|X|}\}$ and $Y = \{y_1, \ldots, y_{|Y|}\}$, where $\forall j \in \{1, \ldots, |X|-1\}$ $x_j P_i x_{j+1}$ and $\forall j \in \{1, \ldots, |Y|-1\}$ $y_j P_i y_{j+1}$ and perform the following comparisons:

1) If $x_1 P_i y_1$, then $X\ EP_i\ Y$.
2) If $x_1 = y_1$ and $|X| < |Y|$, then $X\ EP_i\ Y$.
3) If $x_1 = y_1$ and $|X| = |Y| = k$, where $k \in \{2, \ldots, m-1\}$, then $X\ EP_i\ Y$ is valid iff $x_h P_i y_h$ for the least $h \in \{2, \ldots, k\}$ such that $x_h \neq y_h$

This definition is represented as an algorithm. It can be seen from the formulation itself that all sets are comparable.

**Ordering by Ascending Probability of the Worst Alternative**

This method of extension is similar to the last one to within the order of comparison of the elements. Attention is focused here on the presence of the best result, rather than on the absence of the worst result. Correspondingly, the probability of choosing the worst alternatives is minimized. We describe the method in formal terms and present an example.

For all $i \in N$ and preferences $P_i \in \mathbf{L}$, the extended preferences $EP_i$ are constructed as follows.

The elements of collective choice are ordered in the descending order of preference $P_i$, that is, $X = \{x_1, \ldots, x_{|X|}\}$ и $Y = \{y_1, \ldots, y_{|Y|}\}$, where $\forall j \in \{1, \ldots, |X|-1\}$ $x_j P_i x_{j+1}$ and $\forall j \in \{1, \ldots, |Y|-1\}$ $y_j P_i y_{j+1}$, and the following comparison is carried out:

1) If $x_{|X|} P_i y_{|Y|}$, then $X\ EP_i\ Y$.

2) If $x_{|X|} = y_{|Y|}$ and $|X| > |Y|$, then $X\ EP_i\ Y$ .

3) If $x_{|X|} = y_{|Y|}$ and $|X| = |Y| = k$, where $k \in \{2,\ldots,m-1\}$, then $X\ EP_i\ Y$ is valid iff $x_h P_i y_h$ for the largest $h \in \{2,\ldots,k\}$ such that $x_h \neq y_h$

The extended preferences constructed by this method for three alternatives and standard preferences are as follows:

$$\{a\}EP_i\{a,b\}EP_i\{b\}EP_i\{a,b,c\}EP_i\{a,c\}EP_i\{b,c\}EP_i\{c\}$$

In this case, one can assert that the extended preferences constructed by the probabilistic methods will be the linear orders.

**Method of Averaging Ranks with Additional Constraints**

There is another well-known approach to extending the participant preferences by assigning utility to each alternative and maximizing the expected utility which was proposed by von Neumann and Morgenstern. We define this method for collective choice according to (Pattanaik, 1978).

## *Method of assigning utilities*

For any $i \in N$ there exists a function $U^i(\cdot)$ defined over the set of alternatives and assuming real values such that $\forall \vec{P}, \vec{P}' \in \mathbf{L}^n$

$$C(\vec{P})EP_iC(\vec{P}') \Leftrightarrow \sum_{x\in C(\vec{P})} p_x U^i(x) > \sum_{y\in C(\vec{P}')} p_y U^i(y),$$

where $\forall x \in C(\vec{P})$, $p_x$ is the probability that eventually $x$ will be selected from $C(\vec{P})$ and $\forall y \in C(\vec{P}')$, and $p_y$ is the probability that eventually $y$ will be selected from $C(\vec{P}')$ .

For practical use, this method, obviously, must be completed, that is, it is desired to describe the mechanism of assigning utility and the process of assigning probabilities.

We introduce additional conditions.

1. Utility is assigned to each alternative according to its place in the preferences of the participant. By this we carry out ranking: the most preferable alternative is ranked as $m$, where $m$ is the total number of alternatives, the next, $m-1$ and so on. The least preferable alternative is assigned rank 1.
2. In the collective choice all alternatives are equiprobable. Therefore, the utility of a set of alternatives is equal to the mean utility of all alternatives included in the collective choice.

It is worth noting that this method does not allow to compare all collective choices for $m > 2$. Under standard preferences, for example, for three alternatives there exist sets $\{a,b,c\}$, $\{a,c\}$, and $\{b\}$ that have identical probabilities. It is necessary to consider additional conditions for elimination of incomparability, that is, additional algorithms that would be imposed on the sets remaining incomparable after ranking by an ordinal method.

### *Leximin and Leximax Additions*

This method of elimination of incomparability relies on using lexicographic composition on the set having identical utilities. For the case of three alternatives, in particular, the resulting extended preferences will correspond to the preferences provided by the leximin or leximax rules depending on the method used. On larger sets, however, these two types of extended preferences will not coincide. For example, for the case of four alternatives and standard preferences, the extended leximax-based preferences are as follows:

$$\{a\}EP_i\{a,b\}EP_i\{a,b,c\}EP_i\{a,b,c,d\}EP_i\{a,b,d\}EP_i\{a,c\}EP_i\{a,c,d\}EP_i\{a,d\}EP_i$$
$$EP_i\{b\}EP_i\{b,c\}EP_i\{b,c,d\}EP_i\{b,d\}EP_i\{c\}EP_i\{c,d\}EP_i\{d\}.$$

The extended preferences constructed by rank-averaging with the leximax addition are as follows (underlined are the sets with identical mean rank to which lexicographic addition is applied):

$$\{a\}EP_i\{a,b\}EP_i\underline{\{a,b,c\}EP_i\{a,c\}EP_i\{b\}}EP_i\{a,b,d\}EP_i\underline{\{a,b,c,d\}EP_i\{a,d\}EP_i\{b,c\}}EP_i$$
$$EP_i\{a,c,d\}EP_i\underline{\{b,c,d\}EP_i\{b,d\}EP_i\{c\}}EP_i\{c,d\}EP_i\{d\}.$$

It is obvious that the resulting preferences do not coincide. Similar results are obtained when using the leximin method.

### *Probabilistic Additions*

This method uses the probabilistic additions to the sets that cannot be compared by the method of averaging ranks. For four alternatives and standard preferences, for example, the extended preferences constructed by averaging ranks with additional ranking in ascending probability of the worst alternative are as follows (underlined are the groups of sets with identical mean utility to which addition is applied):

$$\{a\}EP_i\{a,b\}EP_i\underline{\{b\}EP_i\{a,b,c\}EP_i\{a,c\}}EP_i\{a,b,d\}EP_i\underline{\{b,c\}EP_i\{a,b,c,d\}EP_i\{a,d\}}EP_i$$
$$EP_i\{a,c,d\}EP_i\underline{\{c\}EP_i\{b,c,d\}EP_i\{b,d\}}EP_i\{c,d\}EP_i\{d\}$$

### *Risk-lover and Risk-averse: Additions*

This method relies on the attitude of the participant to risk when it is known that the same expected utility defines more than one outcome. Construction of the extended preferences is based on comparing the gains variances. The risk-lover voter will prefer the most risky set, the risk-averse one, the least risky set. Therefore, calculation of the variances and their ascending or descending ranking enables one to eliminate incomparability of the sets with identical mean ranks.

For three, four, five, and six alternatives, the risk-lover addition is similar to ranking by the descending probability of the best alternative, and risk-averse addition is similar to ranking by ascending probability of the worst alternative. Such coincidence is not observed already for seven alternatives.

Let us consider an example. For four alternatives and standard preferences, the method of averaging ranks with the risk-averse addition orders every possible set as follows (underlined are the sets with identical utility to which addition is applied):

$$\{a\} EP_i \{a,b\} EP_i \underline{\{a,c\} EP_i \{a,b,c\} EP_i \{b\}} EP_i \{a,b,d\} EP_i \underline{\{a,d\} EP_i \{a,b,c,d\} EP_i \{b,c\}} EP_i$$

$$EP_i \{a,c,d\} EP_i \underline{\{b,d\} EP_i \{b,c,d\} EP_i \{c\}} EP_i \{c,d\} EP_i \{d\}.$$

### *Addition in the Form of Cardinality Ranking*

The impact of the cardinality of sets on their preference is closely related with the notion of the freedom of choice discussed in (Jones, Sugden, 1982; Pattanaik, Xu, 1990; Sen, 1997). This method lies in ranking the sets which have identical ranks in the descending/ascending cardinality order. It is based on the possible preferences of the participant on the sets with identical expected gain taking into account the definiteness of the resulting choice. For example, in the case of two alternatives with the same ranks, the participant may prefer a one-alternative set to a three-alternative set because in the former case the outcome of voting is known. An opposite situation is possible as well: the participant hopes that the outcome will consist of a more preferable alternative and prefers a set with biggest cardinality to the set with smaller cardinality.

Similar to the previous addition type, for three alternatives coincidence of the results with other methods is observed. In particular, the principle of ordering in ascending cardinality provides for three alternatives the same result as the leximin and the method of averaging ranks with the leximin-addition. The principle of ordering in descending cardinality coincides with the leximax and the method of averaging ranks with the leximax-addition. However, when the number of alternatives increases some problems arise. Apart from non-coincidence of the extended preferences constructed by this method with other methods, incomparability of some sets takes place. It is eliminated by using one of the aforementioned additions.

As an illustration we present extended preferences constructed by the method of averaging ranks with the first addition as ordering by descending cardinality and the second addition being leximax (single underline refers to the sets subjected to the first addition, the double underline denotes the sets remaining incomparable after using the first addition and subjected to the second addition):

$$\{a\}EP_i\{a,b\}EP_i\underline{\{a,b,c\}EP_i\{a,c\}EP_i\{b\}}EP_i\{a,b,d\}EP_i\underline{\{a,b,c,d\}EP_i\underline{\underline{\{a,d\}EP_i\{b,c\}}}}EP_i$$

$$EP_i\{a,c,d\}EP_i\underline{\{b,c,d\}EP_i\{b,d\}EP_i\{c\}}EP_i\{c,d\}EP_i\{d\}.$$

In this case, the extended preferences coincide with the rank leximax, but for five and more alternatives these methods provide different results.

## 4. Manipulability Indices

Two indices are used for the analysis of the degree of manipulability. The first, the so-called Kelly index, was suggested in (Kelly, 1993) as

$$K = \frac{d_0}{(m!)^n},$$

where $d_0$ is the number of all profiles where at least one voter manipulates successfully. Since $m$ is the number of alternatives, $m!$ is the number of all possible linear orders, and since $n$ is the number of the participants, $(m!)^n$ is the number of all possible preference profiles. Therefore, the Kelly index is just a fraction of all manipulable profiles.

In calculations we also use the extended version of the Kelly index proposed in (Aleskerov, Kurbanov, 1999). Let $\lambda_k$ be the number of profiles where precisely $k$ voters can manipulate. In this case, it is possible to construct the index $J_k = \frac{\lambda_k}{(m!)^n}$ showing the number of profiles manipulated precisely by $k$ participants. Obviously, $K = J_1 + J_2 + \ldots + J_n$. Therefore, we use the vector index $J = (J_1, J_2, \ldots, J_n)$

An index of freedom of manipulation was proposed in (Aleskerov, Kurbanov, 1999). In this paper we suggest to use, in addition to the freedom of manipulation, the indices of insensitivity to preference distortions and possibility of worsening the result. Obviously, in the profile $j$ for the participant $i$ there are altogether $m!–1$ different possibilities of preference distortion. Let in $\kappa_{ij}^{+}$ cases a voter by manipulation improves the collective choice for herself as compared with the case of sincere preferences, in $\kappa_{ij}^{0}$ cases the result for her remains the same, and in $\kappa_{ij}^{-}$ cases the final choice will be worse for the voter $i$. Obviously $\kappa_{ij}^{+}+\kappa_{ij}^{0}+\kappa_{ij}^{-}=m!–1$. By dividing each $\kappa_{ij}$ by $m!–1$, we set the corresponding share. By summing the corresponding shares with respect to all participants within one profile and dividing by $n$, we determine the mean share of the corresponding result for the profile. Then, the shares for all profiles are summed in a similar way, and the sum is divided by $(m!)^{n}$. We obtain three indices

$$I_1=\frac{\sum_{j=1}^{(m!)^n}\sum_{i=1}^{n}\kappa_{ij}}{(m!)^n\cdot n\cdot(m!–1)},$$

where $\kappa_{ij}$ is equal to $\kappa_{ij}^{+}$, $\kappa_{ij}^{0}$, or $\kappa_{ij}^{-}$ for determination of the corresponding index. Obviously, $I_1^{+}+I_1^{0}+I_1^{-}=1$.

Whereas the above indices demonstrate the degree of manipulability, the following two indices evaluate an efficiency of manipulation . Let set $C(\vec{P})$ occupying the $k$-th place in the extended preferences of the $i$-th participant be the sincere collective choice. We assume that after manipulation of the $i$-th participant the collective choice becomes $C(\vec{P'})$ occupying the $s$-th place in the extended preferences of the $i$-th participant. Obviously, $s<k$ according to the definition of manipulation. We introduce the value $\theta=k-s$ which shows at what number of “places” in the extended preference the choice of the $i$-th participant is improved. Let us sum $\theta$ for all insincere preferences where manipulation is profitable and divide it by $\kappa_{ij}^{+}$. The resulting index $Z_{ij}$ shows the mean gain in terms of “places” as the result of manipulation of the $i$-th participant in the given profile. Averaging of these indices over all participants within the framework of one profile and over all profiles provides the index

$$I_2=\frac{\sum_{j=1}^{(m!)^n}\sum_{i=1}^{n}Z_{ij}}{(m!)^n\cdot n}.$$

The following index $I_3$ is a modification of $I_2$. The value

$$Z_{ij}^{\max}=\max(\theta_1,...,\theta_{\kappa_{ij}^{+}})$$

is evaluated instead of the estimate $Z_{ij}$ of the mean gain of manipulation of the $i$-th participant. Stated differently, $Z_{ij}^{\max}$ shows the maximum gain in terms of places that can be obtained by the $i$-th participant as the result of manipulation. By averaging in a similar way over all participants and all profiles, we establish the index

$$I_3 = \frac{\sum_{j=1}^{(m!)^n} \sum_{i=1}^{n} Z_{ij}^{\max}}{(m!)^n \cdot n} \ .$$

The indices $I_1$ - $I_3$ were introduced in (Aleskerov, Kurbanov, 1999). The indices $K, I_1, I_2, I_3, J$ are calculated for all rules from the next sections.

## 5. Collective Choice Rules

### *1. Plurality Rule*

The choice includes those alternatives that get more votes than other alternatives, that is,

$$a \in C(\vec{P}) \Leftrightarrow [\forall x \in A \quad n^+(a, \vec{P}) \geq n^+(x, \vec{P})],$$

where $n^+(a, \vec{P}) = card\{i \in N \mid \forall y \in A \ aP_i y\}$

### *2. Approval Voting*

We introduce

$$n^+(a, \vec{P}, q) = card\{i \in N \mid card\{D_i(a)\} \leq q - 1\},$$

that is, $n^+(a, \vec{P}, q)$ is the number of participants for which the alternative $a$ in their preferences is no lower than the $q$-th place. Thus, if $q = 1$, then $a$ is the best alternative for the $i$-th participant; if $q = 2$, then $a$ is either the first or the second best alternative, and so on. The number $q$ is called the level of the procedure.

The approval voting of the level $q$ is defined as

$$a \in C(\vec{P}) \Leftrightarrow [\forall x \in A \ n^+(a, \vec{P}, q) \geq n^+(x, \vec{P}, q)],$$

that is, selected are the alternatives that are among the $q$ best alternatives for the maximum number of the participants.

The approval voting is, obviously, a generalization of plurality rule (the case $q = 1$).

### *3. Borda Rule*

To each alternative $x \in A$ the number $r_i(x, \vec{P})$ is assigned equal to the cardinality of the set of alternatives that are worse than $x$ in the preference $P_i \in \vec{P}$, that is, $r_i(x, \vec{P}) = |L_i(x)| = |\{b \in A : xP_ib\}|$. The sum of these values for $i \in N$ is called the Borda rank for the alternative $x$:

$$r(a, \vec{P}) = \sum_{i=1}^{n} r_i(a, P_i).$$

The choice includes the maximum-rank alternatives

$$a \in C(\vec{P}) \Leftrightarrow [\forall b \in A, \quad r(a, \vec{P}) \geq r(b, \vec{P})].$$

## *4. Black Procedure*

The Condorcet winner, if any, is the collective choice. Otherwise, the Borda rule is used.

## *5. Threshold Rule* (Aleskerov, Yakuba, 2007; Chistyakov, Kalyagin, 2008)

Let $v_1(x)$ be the number of participants for which $x$ is the worst alternative in their preferences, $v_2(x)$ be the number of participants for which $x$ is the second worst alternative, and so on, and $v_m(x)$ be the number of participants for which $x$ is the best alternative. Then the alternatives are ordered lexicographically. The alternative $x$ is said to $V$ - dominate the alternative $y$ if $v_1(x) < v_1(y)$ or there exists $k \leq m$ such that $v_i(x) = v_i(y)$, $i = 1,...,k-1$, and $v_k(x) < v_k(y)$. Stated differently, first of all the numbers are compared of the last places in the orderings for each alternative; if they are equal, then the numbers of the next to the last places are compared, and so on. The alternatives undominated in $V$ represent the choice.

Example. Let us consider the case of five participants and four alternatives. The preferences are the following linear orders:

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
|---|---|---|---|---|
| $a$ | $b$ | $d$ | $a$ | $b$ |
| $c$ | $c$ | $b$ | $c$ | $d$ |
| $d$ | $a$ | $a$ | $d$ | $c$ |
| $b$ | $d$ | $c$ | $b$ | $a$ |

We use this example to demonstrate how the selected method of extension can influence the result of evaluation of manipulability of the voting schemes.

For the rules under consideration, the results of choice for this preference profile are $\{a,b\}$ for the plurality rule, $\{b,c\}$ for the approval rule (for q=2), $\{a,b\}$ for the Borda rule, $\{a,b\}$ for the Black procedure, and $\{c\}$ for the threshold rule.

Let the choice be based on the Borda rule. Then, the result of voting is represented by $\{a,b\}$. We consider the fifth participant in whose preferences the aforementioned alternatives occupy, respectively, the fourth and fifth places. We notice that here the preferences of the fifth participant are not standard, but have the form $bP_5dP_5cP_5a$. Participant 5 can distort the preferences, for example, by interchanging the alternatives $d$ and $c$. In this case, the collective choice goes over to the set $\{a,b,c\}$, and manipulation depends on how the seta $\{a,b,c\}$ and $\{a,b\}$ are preferred. If in each set the alternatives are ordered according to the preferences of the fifth participant, then they go over to $\{b,c,a\}$ and $\{b,a\}$, respectively. Their relation differs in different preferences. For example, in the case of leximax-based preferences arranged in the ascending probability of the worst case, for the fifth participant $\{b,c,a\}EP_5\{b,a\}$, which implies that it is advantageous to participant 5 to distort the preferences in this way. For the rest of methods, such distortion is unprofitable.

## 6. Calculation

The indices were calculated for m = 3, 4, 5 alternatives. For a small number of the participants, all possible preference profiles were examined for manipulability. For larger number of participant, the statistical approach was used.

In both schemes of calculations, m!-1 various "insincere" orderings were generated for each participant within the framework of the considered profile, and the resulting collective choices were compared with the "sincere" choice using the above methods of preference extension.

## 7. Results

Using the above algorithms, four different types of extended preferences can be constructed in the case of three alternativs. For the preferences on a set of alternatives like $aP_ibP_ic$, the following extended preferences were constructed:

1. (**Leximin3**) Leximin, the method of averaging ranks with the leximin addition in ascending power.

$$\{a\}EP_i\{a,b\}EP_i\{b\}EP_i\{a,c\}EP_i\{a,b,c\}EP_i\{b,c\}EP_i\{c\}$$

2. (**Leximax3**) Leximax, the method of averaging ranks with the leximax addition in descending power.

$$\{a\}EP_i\{a,b\}EP_i\{a,b,c\}EP_i\{a,c\}EP_i\{b\}EP_i\{b,c\}EP_i\{c\}$$

3. (**PWorst3**) In increasing probability of the worst, the methos of averaging ranks with the risk-averse addition

$$\{a\}EP_i\{a,b\}EP_i\{b\}EP_i\{a,b,c\}EP_i\{a,c\}EP_i\{b,c\}EP_i\{c\}$$

4. (**PBest3**) In decreasing probability of the best, the method of averaging ranks with the risk-lover addition

$$\{a\}EP_i\{a,b\}EP_i\{a,c\}EP_i\{a,b,c\}EP_i\{b\}EP_i\{b,c\}EP_i\{c\}$$

The figure 3 in the notation of the methods of construction of the extended preferences denotes that the method is used for m=3. In the case of four alternatives these algorithms of preference extension provide 10 different orderings, in the case of five alternatives, 12 orderings. Since the case of three alternatives, obviously, is the most convenient one in terms of result presentation, the following calculations will be carried out mostly for it.

Tables 1 and 2 compile the Kelly indices calculated for voting with three and four participants, respectively. The results obtained by Aleskerov, Kurbanov (1999) who used the alphabetic rule for elimination of incomparability are shown in parentheses near the name of the rule. We notice that in the majority of cases, especially in the case of four participants and unique choice, the level of manipulability was underestimated. It is also seen from the Tables that almost for all rules the Leximin3 and Leximax3 methods give the same values of the Kelly index as PWorst3 and PBest3, respectively.

**Table 1**

**Kelly Index, m=3, n=3**

| | Leximin3 | Leximax3 | PWorst3 | PBest3 |
|---|---|---|---|---|
| Plurality rule (0,1667) | 0,2222 | 0 | 0,2222 | 0 |
| Approval rule q=2 | 0,1111 | 0,6111 | 0,1111 | 0,6111 |
| Borda rule (0,2361) | 0,3056 | 0,4167 | 0,3056 | 0,4167 |
| Black procedure (0,1111) | 0,0556 | 0,1667 | 0,0556 | 0,1667 |
| Threshold rule | 0,3056 | 0,4167 | 0,3056 | 0,4167 |

**Table 2**

**Kelly Index, m=3, n=4**

| | Leximin3 | Leximax3 | PWorst3 | PBest3 |
|---|---|---|---|---|
| Plurality rule (0,1852) | 0,3333 | 0,3333 | 0,3333 | 0,3333 |
| Approval rule q=2 | 0,2963 | 0,2963 | 0,2963 | 0,2963 |
| Borda rule (0,3102) | 0,3611 | 0,4028 | 0,3611 | 0,4028 |
| Black procedure (0,1435) | 0,2361 | 0,2778 | 0,2778 | 0,2361 |
| Threshold rule | 0,4028 | 0,4028 | 0,4028 | 0,4028 |

*Remark.* It deserves special attention the nonmanipulability of the plurality rule for three participants and three alternatives in the case of the Leximax3 and PBest3 methods. This fact may be explained by the fact that all profiles such that all agents have different best alternatives represent a single possible variant of the preference profile where a participant may wish to manipulate and be able to modify the result of voting by distorting the preferences. In this case, each participant has three options:

1. To report sincere preferences. Then, the set $\{a,b,c\}$ will be the result of voting.
2. To move the second best alternative to the upper level of preferences. In this case, it will be the collective choice. For the preferences $aP_ibP_ic$, for example, the choice will be $\{b\}$.
3. To place the worst alternative to the higher level of preferences. In this case, the choice is represented by the worst alternative, $\{c\}$ in this example.

It is obvious that the third option never can be optimal and the decision is about manipulation is made by comparing the sets $\{a,b,c\}$ and $\{b\}$. As can be seen from the methods of preference extension, in the Leximin3 and PWorst3 methods the set $\{b\}$ is superior to $\{a,b,c\}$. Consequently, in this case manipulation takes place. On the contrary, in the Leximax3 and PBest3 methods choice under true preferences is preferable, and there will be no manipulation. That is why the corresponding indices in Table 1 are zero.

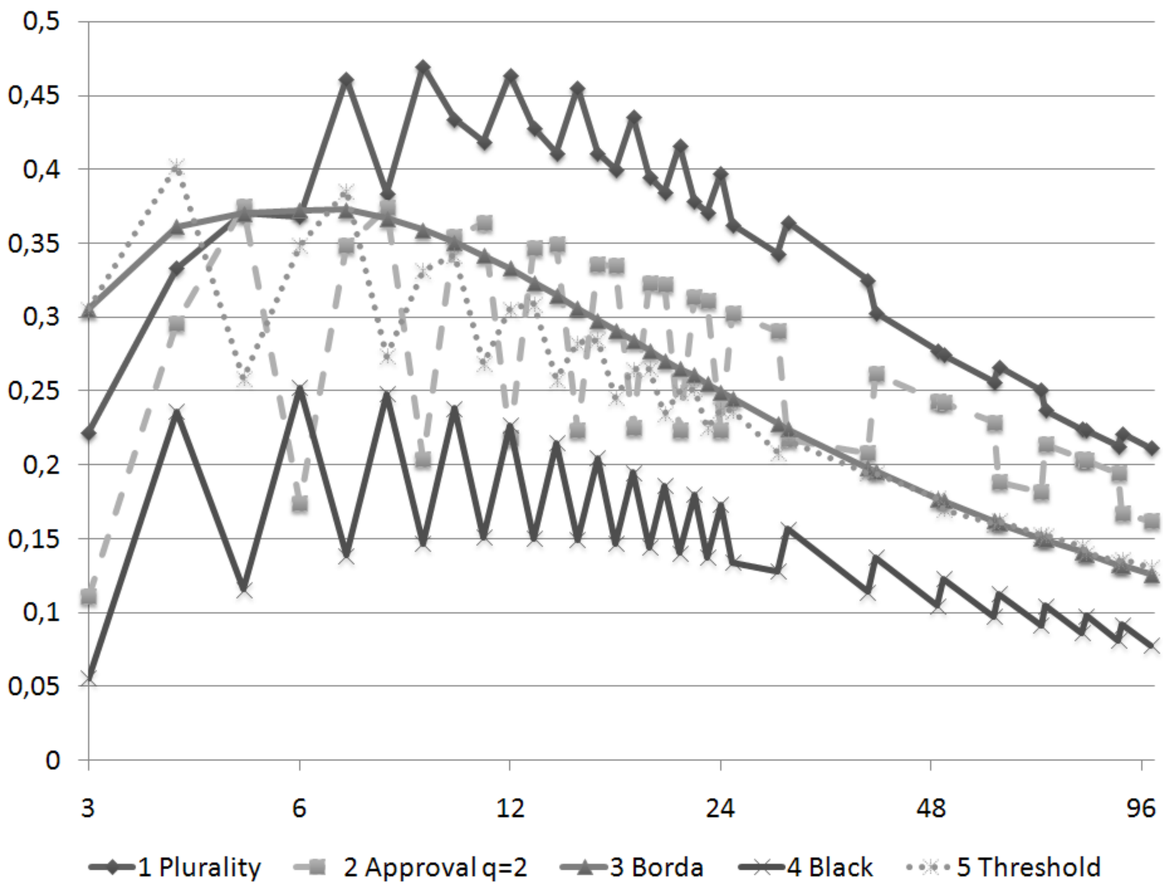


***Fig. 1.*** **Kelly index for the Leximin3 method**

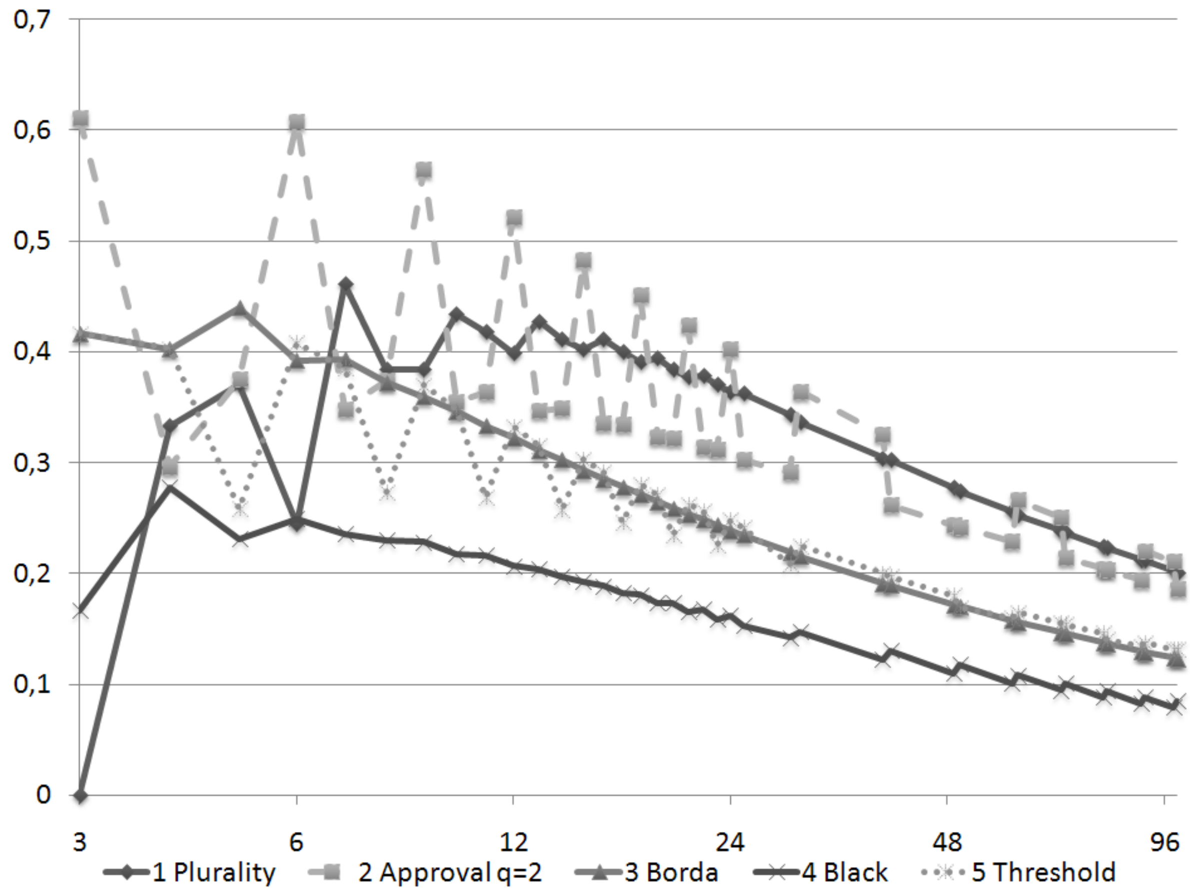


***Fig. 2.*** **Kelly index for the Leximax3 method**

The results of calculations of the Kelly index for a higher number of participants and the Leximin3 and Leximax3 methods are depicted in Figs. 1 and 2, respectively. The value of Kelly index is plotted on the Y axis, and the logarithm of the number of participants, on X. Calculations were carried out for each number of participants from 3 through 25 and then for 29, 30, 39, 40 and so on until 100, which explains the change in behavior of the graphs for more than 25 participants.

These figures suggest some conclusions.

1. The answer to the question of what rule of voting is the least manipulable one depends on the method of preference extension. For example, if the number of participants is a multiple of three, then the approval voting is less manipulable as compared with the plurality rule for the Leximin3 method. For Leximax3 the situation is opposite.

2. For large number of participants, the threshold rule is more manipulable than the Borda rule. The opposite exists almost always for a small number of participants. The number

of participants over which the Borda rule is less manipulable in terms of the Kelly index depends on the method of preference extension used.

3. The Black procedure is the least manipulatable among the considered rules of voting for any method and almost for any number of participants.

4. The Kelly index for the Black procedure and Leximin3 method depends on the parity of the number of participants. At the same time, clearly marked cycles of length $m$ are observed for rules such as the plurality rule, approval voting, and threshold rule. In the case under consideraiton, the cycle has length three. Cyclic dependence of the estimate of manipulability on the number of alternatives is explained by the frequency of occurrence of the multiple choice. For example, the set $\{a,b,c\}$ may result from voting by the plurality rule only if the number of participants is a multiple of the number of alternatives.

These conclusions are cofirmed also in the case where the number of alternatives increases. Fig. 3 shows the results of calculations of the Kelly index for four alternatives and the leximax (Leximax4) method. For the ordinary preferences like $aP_ibP_icP_id$ , the extended Leximax4 preferences are as follows:

$$\{a\}EP_i\{a,b\}EP_i\{a,b,c\}EP_i\{a,b,c,d\}EP_i\{a,b,d\}EP_i\{a,c\}EP_i\{a,c,d\}EP_i\{a,d\}EP_i$$
$$EP_i\{b\}EP_i\{b,c\}EP_i\{b,c,d\}EP_i\{b,d\}EP_i\{c\}EP_i\{c,d\}EP_i\{d\}.$$

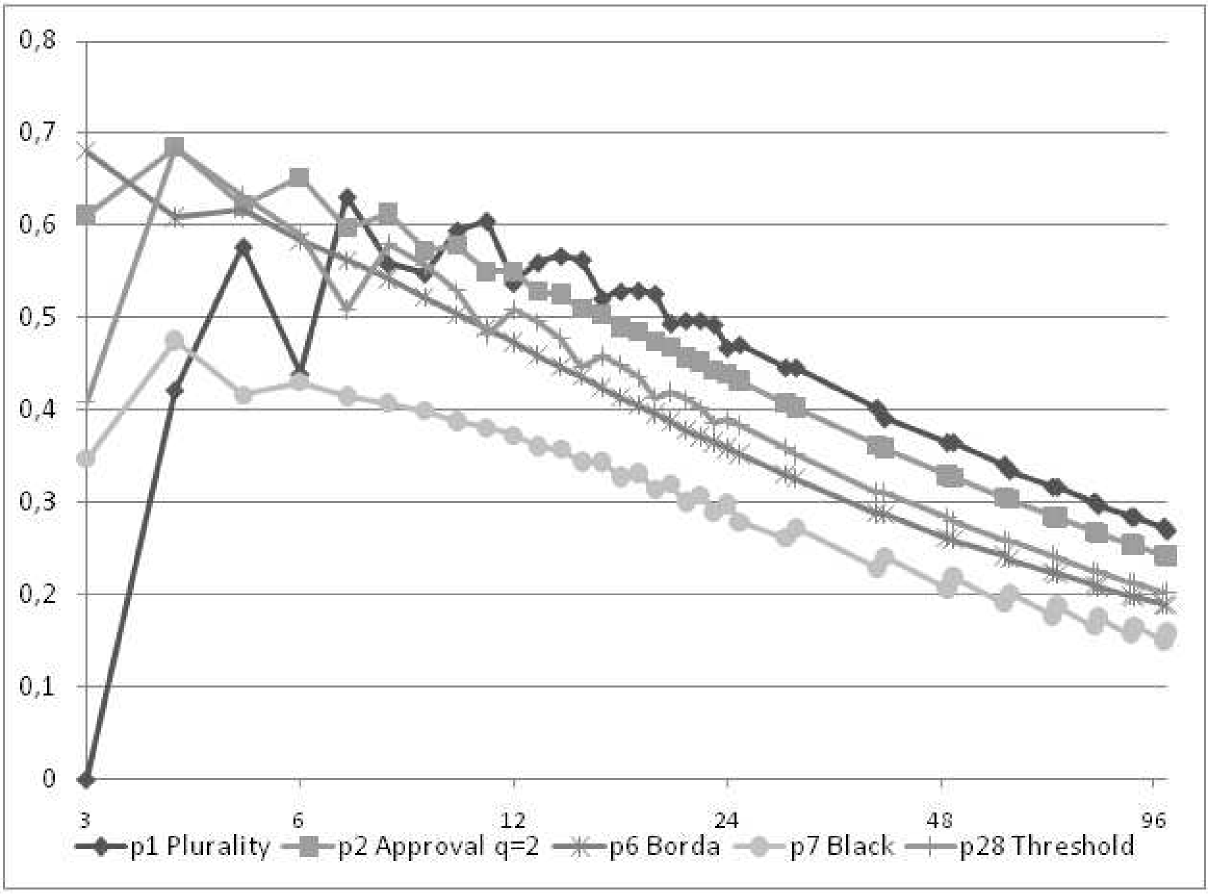


***Fig. 3.*** **Kelly index for the Leximax4 method**

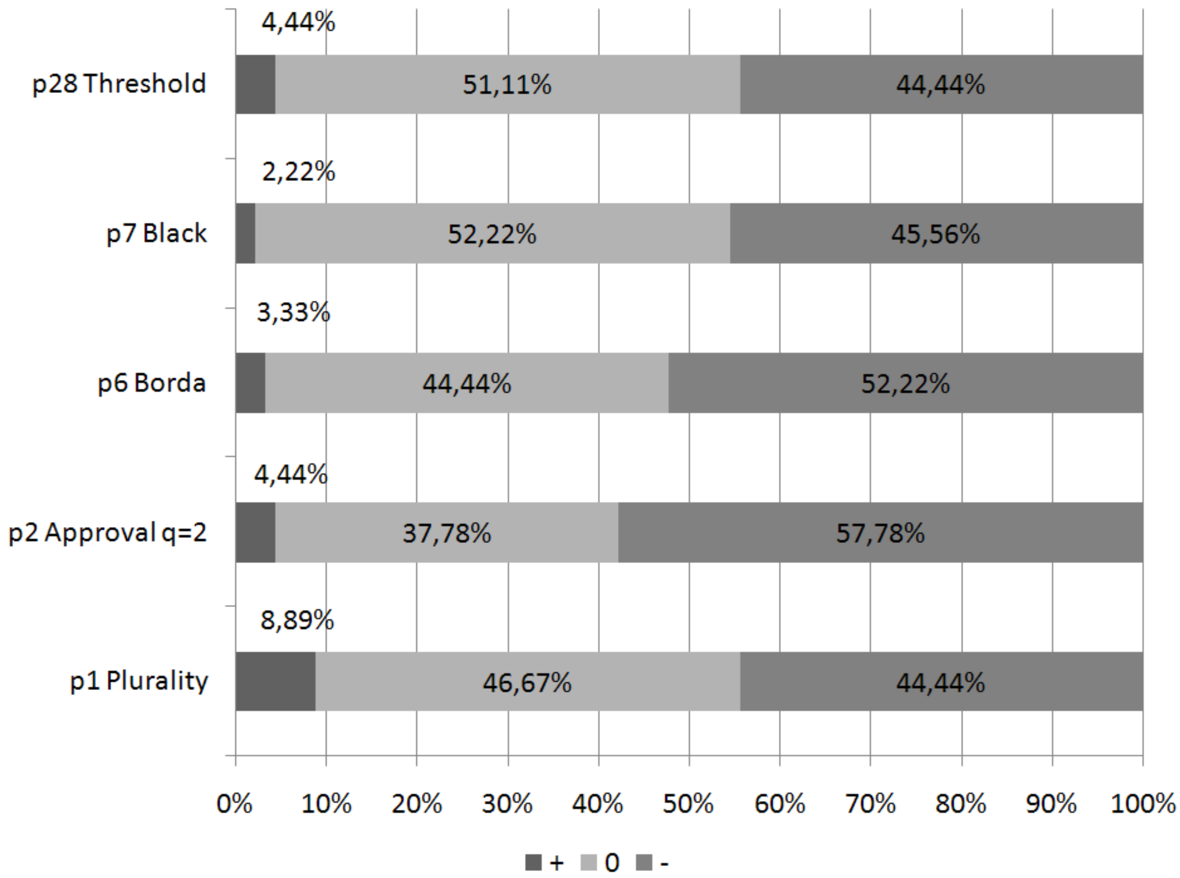


***Fig. 4.*** **$I_1$ for the Leximin3 method, n=3.**

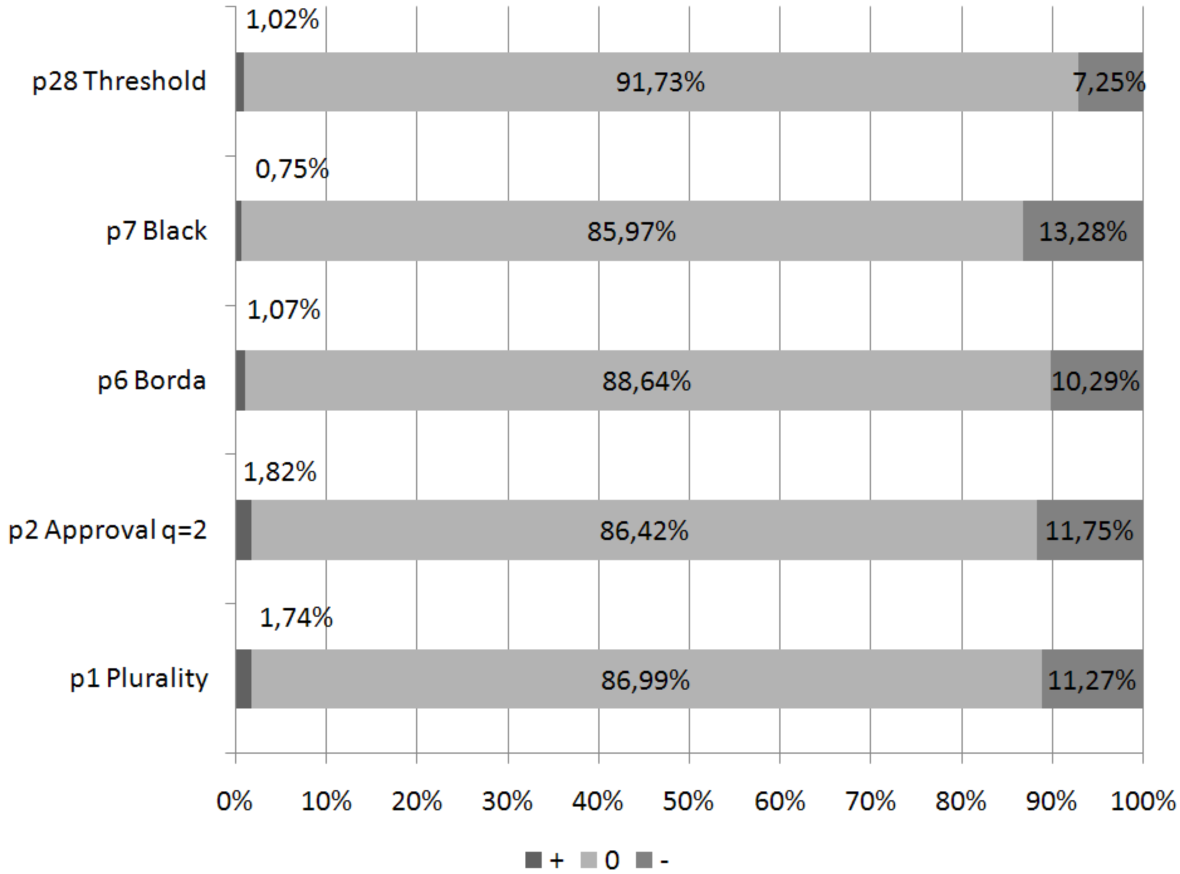


***Fig. 5.*** **$I_1$ for the Leximin3 method, n=100**

As can be seen from Fig. 3, for the plurality rule the Kelly index has cycle of length four. The nonmanipulability of the plurality rule is retained also for the case of three participants.

Fig. 4 shows the index $I_1$ as calculated for three alternatives, three participants and the Leximin3 method. The left part of each row shows the level of manipulation freedom, the right one – the index of possible deterioration, and the middle one, the level of insensitivity to distortions of preferences. Fig. 5 shows the index $I_1$ as calculated for three alternatives, 100 participants, and the Leximin3 method. Notably, the greater the number of participants, the less sensitive the rules of preference distortions, which is in good agreement with the logic of taking collective decisions.

Figs. 6 and 7 depict the results of calculating the indices $I_2$ and $I_3$ for the case of three alternatives and the Leximin3 method.

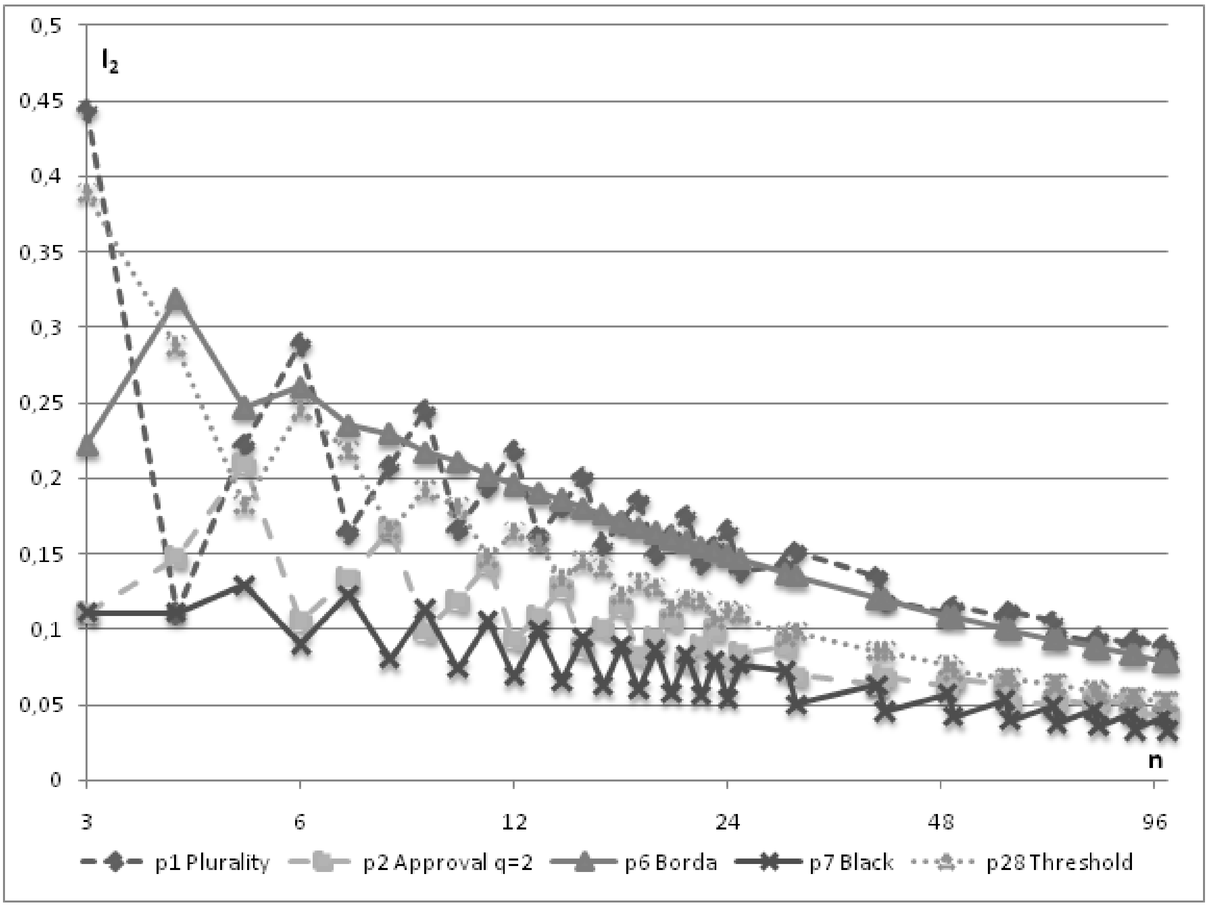


***Fig. 6.*** **$I_2$ for the Leximin3 method**

As can be seen from the figures, efficiency of manipulation, that is, $I_2$**,** decreases with increasing number of the participants, which is due not only to the decreasing possibilities to manipulate, but also to the fact that manipulation itself becomes less efficient. Interestingly, comparison of the rules in terms of the degree of manipulability and freedom of manipulability provides different results. On the one hand, it is possible to see that the Black procedure leads to the least efficiency of manipulation and is the least manipulatable from the point of view of the degree of manipulability. However, the Borda rule, which for a great number of participants was the second after the Black procedure in terms of the Kelly index, in the case of estimation of manipulation effectiveness shows that in the mean the manipulating participants win more with it than with other rules. It also deserves to draw attention to the approval voting which demonstrated low efficiency of manipulation despite a great fraction of the manipulatable profiles.

Comparison of Figs. 6 and 7 shows that in fact there are no differences—namely, the graphs correspoding to the Black procedure and the threshold rule move slightly upward, but the indices for the rest of rules remain the same. One may conclude from this that for m=3 most frequently there is only one possibility of changing the collective choice in the direction preferable for the agent by some insincere preferences.

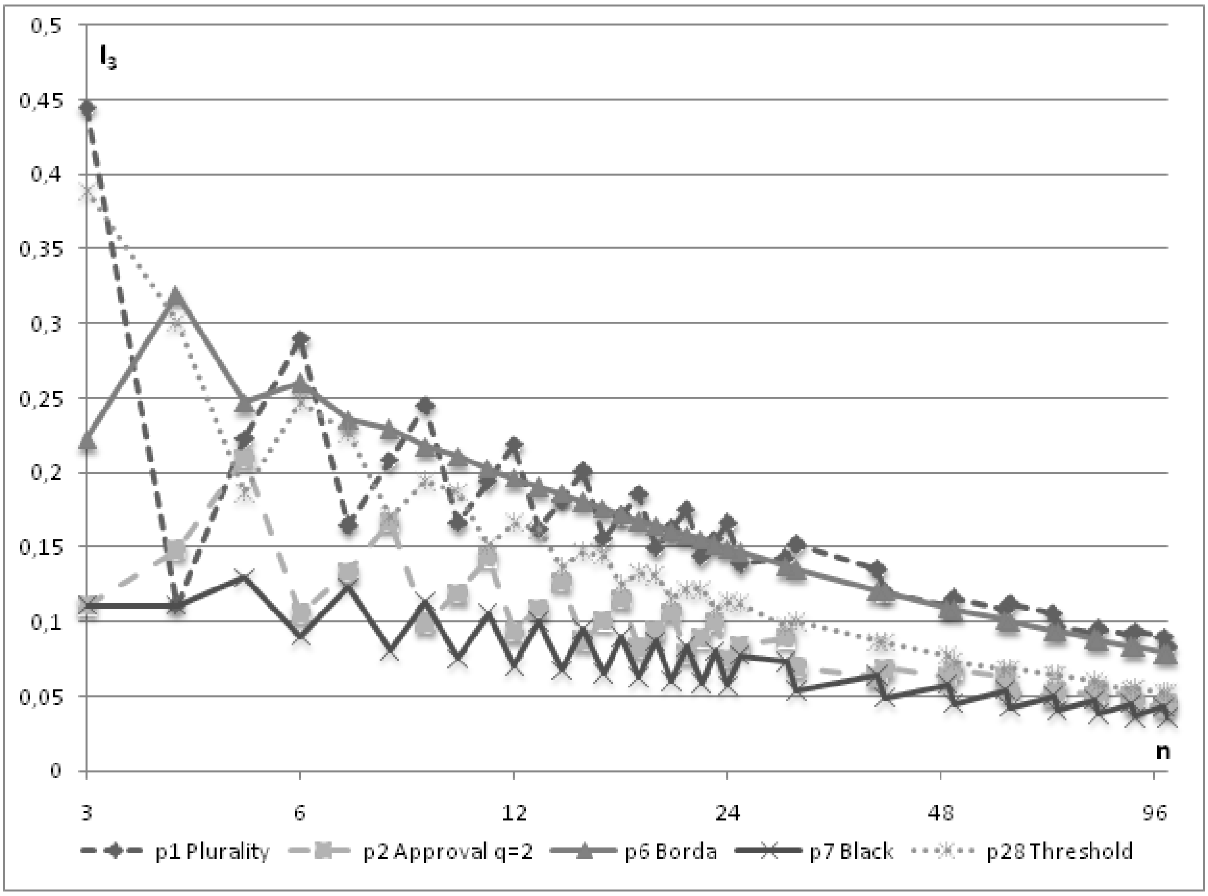


***Fig. 7.*** $I_3$ **for the Leximin3 method**

Until now, we considered the estimate of the voting rule from the point of view of manipulability. Yet it is possible to introduce another, equally important criterion for decisiveness of the rule of voting by which we mean the weight-average number of alternatives in the final set. Therefore, the measure of decisiveness is representable as

$$D = \sum_{\mu=1}^{m} \mu \cdot d_{\mu} \ ,$$

where $d_{\mu}$ is the fraction of the profiles for which the result of voting includes $\mu$ alternatives.

Thus, there are two criteria, the Kelly index and the measure of decisiveness $D$. We can tackle a two-criterion problem of seeking Pareto-efficient voting rules where it is required to minimize the criteria.

Fig. 8 shows "motion" of the voting rules for the number of participants varying as n= 3, 4, 5, 10, 100. Table 3 answers the question of what rules are Pareto-efficient for each number of participants. As can be seen from Fig. 8 and Table 3, the Black procedure is not only the least manipulable one, but also efficient in terms of both criteria in all cases.

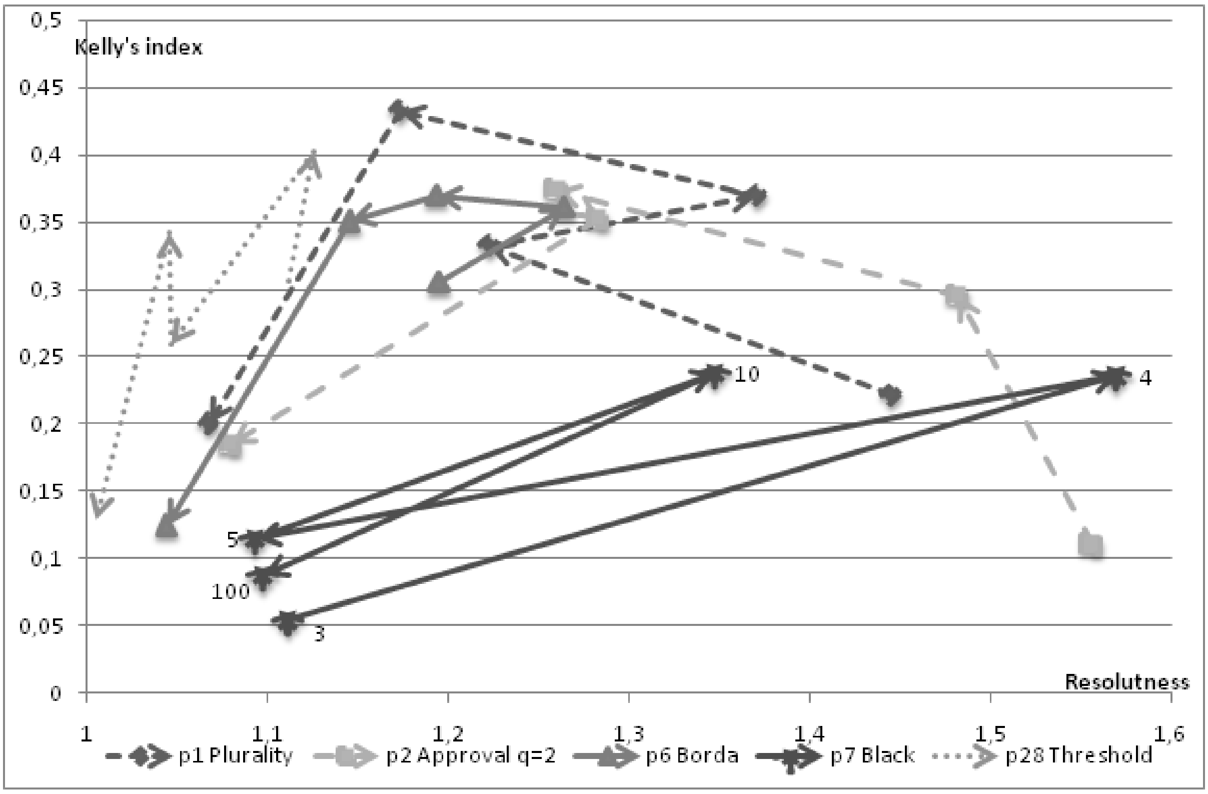


***Fig. 8.*** **Reflection of the "motion" of rules in the space of degree of manipulability for the Leximin3 method at the number of voting participants growng according to the sequence n=3, 4, 5, 10, 100**

**Table 3**

**Pareto-efficient rules for the Leximin3 method and n=3, 4, 5, 10, 100**

***(+ labels such rules for each number of participants)***

| Rule | Plurality rule | Approval voting | Borda | Black | Threshold |
|---|---|---|---|---|---|
| 3 agents | - | - | - | + | - |
| 4 agents | + | + | - | + | + |
| 5 agents | - | - | - | + | + |
| 10 agents | - | - | - | + | + |
| 100 agents | - | - | + | + | + |

## References


**Aleskerov, F.T., Yakuba, V.I.** (2007). A method for threshold aggregation of three-grade rankings // Doklady Mathematics, 75(2): 322-324.

**Aleskerov, F., Kurbanov, E.** (1999). Degree of manipulability of social choice procedures // *Alkan et al. (eds.). Current Trends in Economics*. Berlin: Springer.

**Barbera, S.** (1977). The manipulability of social choice mechanisms that do not leave too much to chance // *Econometrica* 45: 1572-1588.

**Chistyakov, V.V., Kalyagin, V.A.** (2008). A model of noncompensatory aggregation with an arbitrary set of grades // Doklady Mathematics 78(1): 617-620.

**Favardin, P., Lepelley, D.** (2006). Some further results on the manipulability of social choice rules// *Social Choice and Welfare* 26: 485-509.

**Gärdenfors, P.** (1976). Manipulation of social choice functions // *Journal of Economic Theory* 13: 217-228.

**Gibbard, A.** (1973). Manipulation of voting schemes // *Econometrica* 41, 587-601.

**Jones, P. and Sugden, R.** (1982). Evaluating choice// *International Review of Law and Economics* 2: 47–69.

**Kannai Y., Peleg B.** (1984). A note on the extension of an order on a set to the power set // *Journal of Economic Theory* 32: 172-175

**Kelly, J.** (1977). Strategy-proofness and social choice functions without single valuedness // *Econometrica* 45: 439-446

**Kelly, J.** (1993). Almost all social choice rules are highly manipulable, but few aren't // *Social Choice and Welfare* 10: 161-175

**Pattanaik, P.** (1978). Strategy and group choice. Amsterdam: North-Holland.

**Pattanaik, P.K., Peleg B.** (1984). An axiomatic characterization of the lexicographic maximin extension of an ordering over a set to the power set // *Social Choice and Welfare* 1: 113–122.

**Pattanaik, P.K. and Xu, Y.** (1990). On ranking opportunity sets in terms of freedom of choice // *Recherches Economiques de Louvain* 56, 383–390.

**Pritchard, G., Wilson, M.** (2005). Exact results on manipulability of positional voting rules // *CDMTCS Research Report Series*.

**Sanver, R., Ozyurt, S.** (forthcoming). A general impossibility result on strategy-proof social choice hyperfunctions" // *forthcoming in Games and Economic Behavior*.

**Satterthwaite, M.** (1975). Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions // *Journal of Economic Theory* 10: 187-217

**Sen, A.** (1997). Maximization and the act of choice // *Econometrica* 65(4): 745-779.


*Submitted October 6, 2008*